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Dynamical Mass Generation
From Elementary Fields to Bound States

A thesis submitted for the degree of
Doctor of Philosophy
by

Kamel Benhaddou

University of Durham
Department of Physics
October 2003

UNIVERSITY OF DURHAM

FACULTY OF SCIENCE

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Abstract

We investigate the dynamical generation of fermion mass in Quantum Electrodynamics (QED) and in Quantum Chromodynamics (QCD). This non-perturbative study is performed using a truncated set of Schwinger-Dyson equations for the fermion and photon propagator and the quark propagator.

First, we study dynamical fermion mass generation in QED using a cancellation mechanism for the full photon-electron vertex that respects multiplicative renormalisability and reproduces perturbation theory and determine the critical coupling in different approximations. We then study the quark equation using a model for the strong coupling with two parameters and compare this study with previous ones. Finally, we show how bound states masses derived by lattice calculations can be extrapolated to low quark masses using the Nambu Jona-Lasinio model (NJL) and demonstrate the limitation of the NJL model. As an outlook, we present a functional method to control the quantum fluctuations of a given theory. We derive an exact equation for the effective action Γ and using a gradient expansion for Γ we derive evolution equations for different couplings.

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المتنبي

Men are at the dimension of their acts, it is by knowing their acts that we know them.
In the eye of the small, small things are huge, but for the great souls, huge things appear small.

Al Mutanabbi

Abstract

We investigate the dynamical generation of fermion mass in Quantum Electrodynamics (QED) and in Quantum Chromodynamics (QCD). This non-perturbative study is performed using a truncated set of Schwinger-Dyson equations for the fermion and photon propagator and the quark propagator.

First, we study dynamical fermion mass generation in QED using a cancellation mechanism for the full photon-electron vertex that respects multiplicative renormalisability and reproduces perturbation theory and determine the critical coupling in different approximations. We then study the quark equation using a model for the strong coupling with two parameters and compare this study with previous ones. Finally, we show how bound states masses derived by lattice calculations can be extrapolated to low quark masses using the Nambu Jona-Lasinio model (NJL) and demonstrate the limitation of the NJL model. As an outlook, we present a functional method to control the quantum fluctuations of a given theory. We derive an exact equation for the effective action Γ and using a gradient expansion for Γ we derive evolution equations for different couplings.

Declaration

I declare that no material in this thesis has previously been submitted for a degree at this or any other university.

All the research in this work has been carried out in collaboration with Prof. M.R. Pennington.

Acknowledgements

A few lines will be wholly inadequate to express the respect and gratitude owed to those who devoted their time and efforts to ensuring completion of this thesis. I offer but a few names as representatives of the many people who offered support and counsel.

I am especially grateful to the TMR research network council for the trust it put in me when deciding to sponsor my studies in the U.K.. I am most grateful to Prof. Michael Pennington for providing necessary support and assistance and yet allowing me the independence to chart my own path.

I am also grateful for the assistance and cooperation of several people who helped me out along the way simply because I asked. In particular, I would like to thank Jamshidbek Gaziyev for his valuable support.

In addition, I am fortunate to those who provided counsel, offered encouraging words, made me laugh hard, bolstered my spirits at just the right times. Included are, O. Laghrouche, M. Saidi, A. Yahiaoui, Y. Belghitar, M.G.A El Baqir and T. Bakheit. Thanks of each of you for your friendship.

I more than grateful to my parents, Benyacine and Mama, my sisters, Yamina, Samira and Jasmine, and my brothers Muhammad, Nasraddine and Hafid. They were very encouraging and made me strive towards my goals with a greater sense of responsibility and integrity.

Lastly, I would like to thank the One who makes everything possible.

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Chapter 1

Introduction

Despite the appeal, interest and promise of string theory, Quantum Field Theory (QFT) has been more phenomenologically successful in explaining the physical properties of particles which make up our knowledge of the universe. The so-called standard model of strong, weak and electromagnetic interactions is an achievement whose success continues to be revered. The only drawbacks one could point at are the use of the hypothetical Higgs boson and the number of parameters needed to fix the masses of the gauge bosons and leptons. In a classical treatment, the mass is just a parameter which has to be measured experimentally and used in models to describe the particles and their interactions. Obviously if we are looking for a theory of everything, then we must find an answer to the question of the origin of mass. Even if we consider a Yukawa coupling between the Higgs and each fermion, generating mass for the fermions, we still have to resolve the problem of fine tuning to keep the fermion masses at the scale at which they are experimentally measured. The concept of dynamical symmetry breaking is attractive since it offers the possibility of mass generation without relying on the existence of a scalar field. Perturbative corrections are always proportional to the bare mass, so if we start with a vanishing seed, we cannot generate a non zero mass. Dynamical mass generation is therefore essentially a non-perturbative phenomenon. Schwinger-Dyson Equations (SDE) and lattice gauge theory are by now considered old subjects and both are non-perturbative techniques, each one with its own advantages and drawbacks. Schwinger-Dyson equations are derived in the continuum in either Minkowski or Euclidean space and they can be studied for any value of the bare quark mass. Their limitations stem from the fact that they are a system of integral equation that are infinite in number. Each n-point function satisfies an integral equation which involves an m-point function of higher order i.e. $m > n$. In practice, we have to resort to a truncation scheme, which makes the prediction both scheme and gauge dependent. Lattice gauge theory is a discrete theory which in principle is valid for any value of the bare quark mass. However, the computer limitations of Monte Carlo simulations precludes the simulation of light quarks. On one hand (SDE) we have to truncate the theory and use the right parameters, while on the other hand we can use the full theory but with non-physical parameters. The ultimate goal of this thesis is to try to reconcile these two conflicting approaches.

SDE have already been solved for different field theories in various truncation schemes. The most evident one is to approximate the full fermion-gauge boson vertex by its bare value. This simplifies substantially the equations at the price of introducing non-physical artefacts, since this vertex does not satisfy the underlying gauge symmetry as expressed in the Ward-Takahashi (WT) or Slavnov-Taylor identities (STI) or the multiplicative renormalisability of the theory. Improvements have of course been made, by trying to construct vertices that satisfy the STI, but these approaches are cumbersome since it complicates the equations. In our study, we will use a truncation scheme, first introduced by Bloch [4], which reproduces the perturbation theory results, satisfies the important property of multiplicative renormalisability, believed to hold non-perturbatively, and spares us the construction of a complicated full vertex. We will therefore study this method and its predictions for renormalised QED in four dimensions. After determining the critical values for dynamical mass generation in QED_{3+1}, we treat the more interesting case of the quark equation in QCD, where the coupling function is modelled by the use of two parameters, one to fix the infra-red value of the coupling and the second to fix the behaviour at intermediate momenta. We solve this equation in a way that leads to a mass function that is explicitly independent of the renormalisation scale μ^2, i.e. does not need cancellation between the μ^2 dependent terms as is found in usual treatments. Once the quark propagator is known, it can be used to solve the integral equation satisfied by the Bethe-Salpeter (BS) amplitude describing a meson bound state. Unfortunately, the BS equation requires analytical continuation of the quark propagator into the complex domain and this task is too difficult for the moment when one uses complicated models for the coupling function. We will therefore use a model which is simple yet able to capture the essential features of QCD, i.e. the Nambu Jona-Lasinio model (NJL), which enables us to write the mass of the pseudoscalar and vector meson as a function of two couplings, a bare quark mass and a cut-off. Though meson masses depend on the bare quark mass in a complicated manner, they do so continuously. It is therefore here that we will try to make contact with lattice gauge theory, which is reliable only for high quark mass. Using the NJL model, we will try to reproduce the lattice predictions which is a way of testing the validity of the NJL model itself. While its prediction will agree with the lattice calculation, we will show that the claim that the NJL is able to describe the properties of the ρ meson seems to be fortuitous. We nevertheless show that the NJL model incorporates the right behaviour for the pion and can therefore be used to extrapolate lattice calculations to small quark mass, which is better than using empirical fitting formulae as is usually done.

Finally, we will introduce a recent technique to calculate the effective action by varying a parameter of the Lagrangian. We will illustrate the method for the case of a four fermion interaction and derive exact and approximate relations between the couplings. Even though we do not explicitly solve these equations or apply this method to the more interesting NJL model, we demonstrate its potential.

Chapter 2

Schwinger-Dyson Equations

Schwinger-Dyson equations are a non-perturbative tool, namely a set of integral equations for the n-point functions. This set of equations is infinite and thus need to be truncated to be treated in practice. Such truncation schemes are not all consistent, and it is still a challenge to find such a scheme, which would provide a proper understanding of non-perturbative physics. Notwithstanding this fact, there have been many attempts to solve this system of equations in different truncation schemes, which have shed light on the physics of confinement, dynamical mass generation and many other topics [1, 2]. In this chapter, we will give an introduction to the topic of Schwinger-Dyson equations [1] for QED and present some truncation schemes which have been applied to study dynamical symmetry breaking in QED [3, 4].

2.1 QED

The Lagrangian for QED in $d = D + 1$ dimensions is given by

$$\mathcal{L}_{QED} = \sum_{f=1}^{N_f} \bar{\psi}^f \left(i \not{\partial} - m_0^f + e_0^f \not{A} \right) \psi^f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.1}$$

and is the basis of our study. The index f denotes the flavour of the fermion field $\psi^f(x)$ with mass m_0^f and which couples to the photon field A_μ associated to the field strength $F^{\mu\nu}$ defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{2.2}$$

The gauge invariance of the Lagrangian $\mathcal{L}_{QED}$, is assured by giving the fields $A_\mu(x)$ and $\psi(x)$ the following transformation properties

$$\begin{aligned}\psi(x) \to {}^\lambda\psi(x) &= e^{-ie_0\lambda(x)}\psi(x) \;,\;\; \bar{\psi}(x) \to {}^\lambda\bar{\psi}(x) = e^{ie_0\lambda(x)}\bar{\psi}(x) \;, \\ A_\mu(x) \to {}^\lambda A_\mu(x) &= A_\mu(x) - \partial_\mu\lambda(x) \end{aligned} \tag{2.3}$$

Quantisation of $\mathcal{L}_{QED}$ is achieved by the functional integral $\mathcal{Z}[\bar{\eta}, \eta, J_\mu]$ defined as

$$\mathcal{Z}[\bar{\eta}, \eta, J_\mu] = \int D\left[\bar{\psi}, \psi, A_\mu\right] \exp^{\left(iS[\bar{\psi},\psi,A_\mu]+i\int d^{\mathrm{d}}x\,\left[\sum_f\left(\bar{\psi}^f\eta^f+\bar{\eta}^f\psi^f\right)+A_\mu J^\mu\right]\right)}, \tag{2.4}$$

where $\bar{\eta}^f$, η^f and J_μ are the source fields for the fermion, antifermion and gauge boson, respectively, and where we have defined the measure

$$D\left[\bar{\psi}, \psi, A_\mu\right] = \prod_f \mathcal{D}\bar{\psi}^f\mathcal{D}\psi^f \prod_\mu \mathcal{D}A_\mu. \tag{2.5}$$

The normalisation of the functional $\mathcal{Z}[\bar{\eta}, \eta, J_\mu]$, is set by fixing the value of $\mathcal{Z}$, for vanishing sources

$$\mathcal{Z}[0, 0, 0] = 1. \tag{2.6}$$

Because of the gauge invariance of the Lagrangian $\mathcal{L}_{QED}$, the functional $\mathcal{Z}[\bar{\eta}, \eta, J_\mu]$ is for the moment only formally defined. Indeed the integration measure $D\left[\bar{\psi}, \psi, A_\mu\right]$ does not discriminate between the different fields $\bar{\psi}^f$, ψ^f and A_μ, and their gauge transform ${}^\lambda\bar{\psi}$, ${}^\lambda\psi$, and ${}^\lambda A_\mu$, thus giving an infinite value to the functional Z. In order to avoid integrating over fields that are equal up to a gauge transformation, one has to select one equivalent class of these fields. This is achieved by adding to the Lagrangian a non-gauge invariant term that selects only one orbit of the fields. The usual choice for this is the covariant gauge fixing term, which gives

$$S[\bar{\psi}, \psi, A_\mu] \to S_{\xi_0}[\bar{\psi}, \psi, A_\mu] = S[\bar{\psi}, \psi, A_\mu] - \frac{1}{2\xi_0}\int d^{\mathrm{d}}x\; (\partial_\mu A^\mu)^2 \;, \tag{2.7}$$

where ξ_0 is the bare gauge fixing parameter.

The functional $\mathcal{Z}[\bar{\eta}, \eta, J_\mu]$ is the generating functional for the n-point functions G_n, defined as

$$G_n(x_1, x_2, \ldots, x_n) = \left(\frac{1}{i}\right)^n \frac{\delta^n}{\delta F_1 \delta F_2 \ldots \delta F_n}\, \mathcal{Z}[\bar{\eta}, \eta, J_\mu]|_{\bar{\eta}=\eta=j^\mu=0}\,, \tag{2.8}$$

where F_i represent any of the fields $\bar{\psi}$, ψ or A_μ. These n-point functions are non-local in space and it is thus better to introduce the connected functions $G^c_n(x_1, x_2, \ldots, x_n)$ generated by the functional $\mathcal{W}[\bar{\eta}, \eta, J_\mu]$ defined as

$$\mathcal{Z}[\bar{\eta}, \eta, J_\mu] = e^{\mathcal{W}[\bar{\eta}, \eta, J_\mu]}. \tag{2.9}$$

These connected n-point functions still involve graphs that contains fermion or photon internal lines. These can be made more localised in space by introducing the generating functional for proper graphs $\Gamma[\bar{\psi}, \psi, A_\mu]$, which is the Legendre transform of $\mathcal{W}[\bar{\eta}, \eta, J_\mu]$

$$i\Gamma[\bar{\psi}, \psi, A_\mu] = \mathcal{W}[\bar{\eta}, \eta, J_\mu] - i \int d^d x \, \left(\bar{\psi}\eta + \bar{\eta}\psi + A_\mu J^\mu\right). \tag{2.10}$$

The arguments of the effective action Γ are the fermion fields and the photon field $\bar{\psi}$, ψ and A_μ, respectively defined by the relations

$$\begin{aligned}
\bar{\psi}(x) &= \frac{1}{i}\mathcal{W}\frac{\overleftarrow{\delta}}{\delta\eta} \\
\psi(x) &= \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\mathcal{W} \\
A_\mu(x) &= \frac{1}{i}\frac{\delta}{\delta J^\mu}\mathcal{W}.
\end{aligned} \tag{2.11}$$

The relations define the fields as functions of the sources and are implicitly used to defined the effective action Γ, which is thus only a function of the fields $\bar{\psi}$, ψ and A_μ as defined in Eq. (2.11). The definition of the effective action Γ of Eq. (2.10) permits us to write the following relations

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_\mu} &= -j^\mu\,, \\
\frac{\delta\Gamma}{\delta\bar{\psi}} &= -\eta\,, \\
\Gamma\frac{\overleftarrow{\delta}}{\delta\psi} &= -\bar{\eta}\,, \\
\frac{\delta}{\delta\bar{\psi}}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi} &= -\frac{\delta\bar{\eta}}{\delta\bar{\psi}} = -i\left(\frac{\delta}{\delta\bar{\eta}}W\frac{\overleftarrow{\delta}}{\delta\eta}\right)^{-1}.
\end{aligned} \tag{2.12}$$

The connected 2-point fermion Green's function or *fermion propagator* $S(x,y)$ is

$$S^{-1}_{ab}(x,y) = \frac{\delta}{\delta\bar{\psi}_b(y)}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi_a(x)}\Bigg|_{\psi,\bar{\psi},A_\mu=0} . \tag{2.13}$$

We define the connected 2-point photon Green's function or *photon propagator* $D^{\mu\nu}(x,y)$ as

$$D^{-1}_{\mu\nu}(x,y) = \frac{\delta^2\Gamma}{\delta A^\nu(y)\delta A^\mu(x)}\Bigg|_{\psi,\bar{\psi},A_\mu=0} . \tag{2.14}$$

The 1PI 3-points Green's function or *vertex* $e_0\Gamma(x,y;z)$ is defined by

$$e_o\Gamma^\mu_{ab}(x,y;z) = -\frac{\delta^2}{\delta A_\mu(z)\delta\bar{\psi}_a(x)}\Gamma\frac{\overleftarrow{\delta}}{\delta\psi_b(y)}\Bigg|_{\psi,\bar{\psi},A_\mu=0} . \tag{2.15}$$

The Schwinger-Dyson equations can be derived by applying the functional integral formalism to the *QED* Lagrangian. They correspond to the Euler-Lagrange equation for the quantum field theory defined by the Lagrangian $\mathcal{L}_{QED}$ of Eq. (2.1).

2.2 Unrenormalised Equation for the Photon Polarisation Tensor

In order to obtain the Schwinger-Dyson Equations (SDEs) one has just to remember that, the functional integral of a total functional derivative is zero, with fields vanishing at infinity. Hence, for example [1]

$$\begin{aligned} 0 &= \int D\left[\bar{\psi},\psi,A_\mu\right]\frac{\delta}{\delta A_\mu(x)}\exp^{\left\{i\left(S_\xi[\bar{\psi},\psi,A_\mu]+\int d^dx\,\left[\bar{\psi}^f\eta^f+\bar{\eta}^f\psi^f+A_\mu J^\mu\right]\right)\right\}} \\ &= \int D\left[\bar{\psi},\psi,A_\mu\right]\left\{\frac{\delta S_\xi}{\delta A_\mu(x)}+J_\mu(x)\right\}\exp^{\left\{i\left(S_\xi[\bar{\psi},\psi,A_\mu]+\int d^dx\,\left[\bar{\psi}^f\eta^f+\bar{\eta}^f\psi^f+A_\mu J^\mu\right]\right)\right\}} \\ &= \left\{\frac{\delta S_\xi}{\delta A_\mu(x)}\left[\frac{\delta}{i\delta J},\frac{\delta}{i\delta\bar{\eta}},\frac{\overleftarrow{\delta}}{i\delta\eta}\right]+J_\mu(x)\right\}\mathcal{Z}[\bar{\eta},\eta,J_\mu] . \end{aligned} \tag{2.16}$$

Differentiating the gauge fixed action Eq. (2.7) with respect to A_μ gives

$$\frac{\delta S_\xi}{\delta A_\mu(x)} = \left[\partial_\rho\partial^\rho g_{\mu\nu}-\left(1-\frac{1}{\xi_0}\right)\partial_\mu\partial_\nu\right]A^\nu+\sum_f e_0^f\,\bar{\psi}^f\gamma_\mu\psi^f , \tag{2.17}$$

which help us to rewrite Eq. (2.16) as

$$
\begin{aligned}
-J_\mu(x) &= \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\xi_0}\right)\partial_\mu\partial_\nu\right]\frac{\delta\mathcal{W}}{i\delta J_\nu(x)} \\
&+\sum_f e_0^f\left(\frac{\delta\mathcal{W}}{\delta\eta^f(x)}\gamma_\mu\frac{\delta\mathcal{W}}{\delta\bar{\eta}^f(x)}+\frac{\delta}{\delta\eta^f(x)}\left[\gamma_\mu\frac{\delta\mathcal{W}}{\delta\bar{\eta}^f(x)}\right]\right). \qquad (2.18)
\end{aligned}
$$

Using the relations involving the effective action Γ in Eq. (2.12) and setting to zero the sources $\bar{\psi}$ and ψ, we can write

$$
\begin{aligned}
\left.\frac{\delta\Gamma}{\delta A^\mu(x)}\right|_{\psi=\bar{\psi}=0} &= \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\xi_0}\right)\partial_\mu\partial_\nu\right]A^\nu(x) \\
&-i\sum_f e_0^f\mathrm{Tr}\left[\gamma_\mu S^f(x,x,[A_\mu])\right]. \qquad (2.19)
\end{aligned}
$$

We differentiate this equation one more time with respect to $A_\nu(y)$ and set the source $J_\mu(x)=0$. We also use the proper fermion-photon vertex defined in Eq. (2.15) and obtain the equation satisfied by the photon propagator

$$
\left[D^{-1}\right]^{\mu\nu}(x,y) = \left[\partial_\rho\partial^\rho g_{\mu\nu} - \left(1-\frac{1}{\xi_0}\right)\partial_\mu\partial_\nu\right]\delta^{\mathrm{d}}(x-y)+\Pi_{\mu\nu}(x,y). \qquad (2.20)
$$

This equation involves the photon polarisation tensor, $\Pi_{\mu\nu}$ defined as

$$
\Pi_{\mu\nu}(x,y) = i\sum_f (e_0^f)^2\int d^{\mathrm{d}}z_1\, d^{\mathrm{d}}z_2\,\mathrm{Tr}\left[\gamma_\mu S^f(x,z_1)\Gamma_\nu^f(y;z_1,z_2)S^f(z_2,x)\right]. \qquad (2.21)
$$

In momentum space, the SD equation for the photon is

$$
\left[D^{-1}\right]^{\mu\nu}(q) = -q^2\left[g^{\mu\nu}+\left(\frac{1}{\xi_0}-1\right)\frac{q^\mu q^\nu}{q^2}\right]+\Pi^{\mu\nu}(q), \qquad (2.22)
$$

with

$$
\Pi^{\mu\nu}(q) = \frac{iN_f e^2}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\left[\gamma^\mu\, S(k)\,\Gamma^\nu(k,k-q)\,S(k-q)\right]. \qquad (2.23)
$$

Because of the Ward-Takahashi identity [WTI] for the photon propagator,

$$
q_\mu\Pi^{\mu\nu}(q) = 0, \qquad (2.24)
$$

we can factor $\Pi^{\mu\nu}(q)$ in the following way

$$
\Pi^{\mu\nu}(q) = (-q^2 g^{\mu\nu}+q^\mu q^\nu)\Pi(q^2), \qquad (2.25)
$$

where $\Pi(q^2)$ is a general function. We are thus able to write the full inverse photon propagator as

$$\left[D^{-1}\right]^{\mu\nu}(q) = -q^2\left[\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\left(1+\Pi(q^2)\right) + \frac{1}{\xi}\frac{q^\mu q^\nu}{q^2}\right]. \qquad (2.26)$$

The inversion of the previous expression yields the full photon propagator, which is related to the inverse propagator in the following way

$$D_{\mu\lambda}(q)\left[D^{-1}\right]^{\lambda\nu}(q) = g_\mu^\nu. \qquad (2.27)$$

The most general tensor expression for the propagator is

$$D_{\mu\nu}(q) = Ag_{\mu\nu} + q^\mu q^\nu/q^2. \qquad (2.28)$$

If we use it in Eq. (2.27), we can determine the two terms A and B. In this way the full propagator has the form

$$D_{\mu\nu}(q) = -\frac{1}{q^2}\left[G(q^2)\left(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi_0\frac{q^\mu q^\nu}{q^2}\right], \qquad (2.29)$$

where $G(q^2)$ is the photon renormalisation function defined as

$$G(q^2) = \frac{1}{1+\Pi(q^2)}. \qquad (2.30)$$

This general expression for the full propagator permits us to derive the expression of the bare propagator, by setting the function $G(q^2)$ to unity or $\Pi(q^2)$ to zero

$$D^0_{\mu\nu}(q) = -\frac{1}{q^2}\left[\left(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi_0\frac{q^\mu q^\nu}{q^2}\right], \qquad (2.31)$$

as is expected.

The SDE for the photon propagator is represented diagrammatically in Fig. (2.1). The momentum-space representation of the SDEs is readily obtained by either Fourier transforming the coordinate-space form or by generalising the standard rules for Feynman diagrams based on the lowest-order perturbative contribution to the nonperturbative quantities. For example, for the photon polarisation tensor we obtain

$$i\Pi_{\mu\nu}(q) = (-1)\sum_f (e_0^f)^2\int\frac{d^{\mathrm{d}}k}{(2\pi)^{\mathrm{d}}}\mathrm{Tr}[(i\gamma_\mu)(iS^f(k))(i\Gamma^f(k,k+q))(iS^f(k+q))]\,, \qquad (2.32)$$

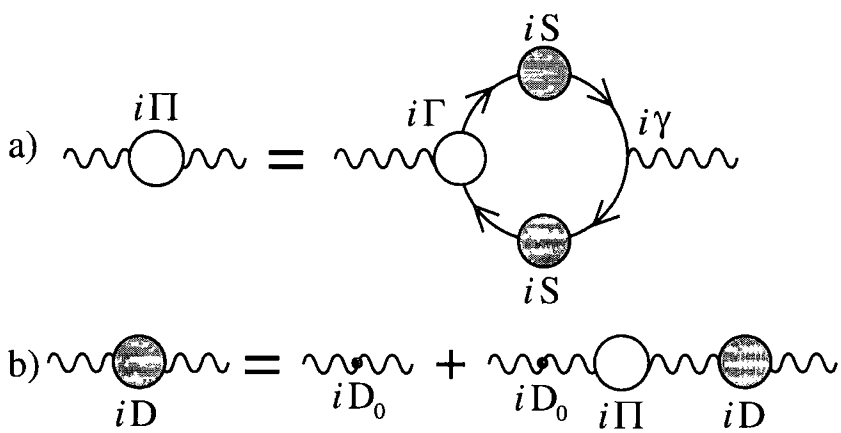

Figure 2.1: The Schwinger-Dyson equation for the photon propagator [1]

where the factor of (-1) arises from the fermion loop in the usual way. In momentum space, Eq. (2.20) corresponds to

$$iD^{\mu\nu}(q) = iD_0^{\mu\tau}(q)[\delta_\tau^\nu + i\Pi_{\tau\rho}(q)iD^{\rho\nu}(q)], \tag{2.33}$$

which can also be obtained from the Fourier transform of Eq. (2.21).

2.3 Unrenormalised Equation for the Fermion Self Energy

We derive the equation satisfied by the self energy (inverse propagator) of the fermion by following very similar steps to those in the previous section. We can write [1]

$$\begin{aligned} 0 &= \int D\left[\bar{\psi}, \psi, A_\mu\right] \frac{\delta}{\delta\bar{\psi}(x)} \exp^{\left\{i\left(S_\xi[\bar{\psi},\psi,A_\mu]+\int d^{\mathrm{d}}x\,\left[\bar{\psi}^f\eta^f+\bar{\eta}^f\psi^f+A_\mu J^\mu\right]\right)\right\}} \\ &= \left\{\frac{\delta S_\xi}{\delta\bar{\psi}(x)}\left[\frac{\delta}{i\delta J}, \frac{\delta}{i\delta\bar{\eta}}, -\frac{\delta}{i\delta\eta}\right] + \eta(x)\right\} \mathcal{Z}[\bar{\eta}, \eta, J_\mu] \ . \end{aligned} \tag{2.34}$$

After differentiating with respect to η and setting all sources to zero ($\bar{\eta} = \eta = J^\mu = 0$) we can rewrite Eq. (2.34) as

$$\begin{aligned} \delta^{\mathrm{d}}(x-y) &= (i\,\partial\!\!\!/ - m_0^f)\, S^f(x,y) \\ &\quad -i(e_0^f)^2 \int d^{\mathrm{d}}z_1\, d^{\mathrm{d}}z_2\, d^{\mathrm{d}}z_3\, \gamma_\mu D^{\mu\nu}(x,z_1) S^f(x,z_2)\Gamma_\nu^f(z_1;z_2,z_3)S^f(z_3,y), \end{aligned} \tag{2.35}$$

where $D_{\mu\nu}(x,y)$ and $\Gamma^{\mu}(x,y,z)$ are the photon propagator and the proper vertex defined in Eq. (2.14) and Eq. (2.15), respectively. We rewrite this equation in terms of the fermion self energy, $-i\Sigma^f(x,y)$, defined such that

$$(i\,\partial\!\!\!/ - m_0^f)\,S^f(x,y) - \int d^{\rm d}z_1\,\Sigma^f(x,z_1)\,S^f(z_1,y) = \delta^{\rm d}(x-y), \tag{2.36}$$

with

$$-i\Sigma^f(x,y) = (e_0^f)^2 \int d^{\rm d}z_1\,d^{\rm d}z_2\,\gamma_\mu D^{\mu\nu}(x,z_1)S^f(x,z_2)\Gamma_\nu^f(z_1;z_2,y). \tag{2.37}$$

The equation for the fermion self-energy is represented diagrammatically in Fig. (2.2).

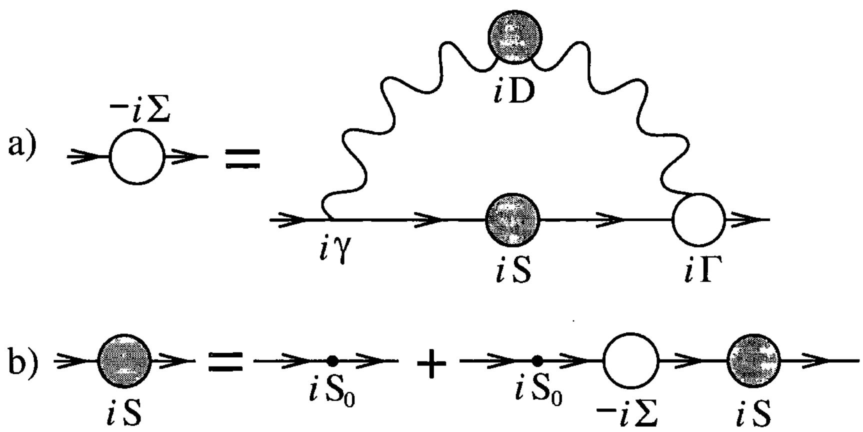

Figure 2.2: The Schwinger-Dyson equation for the electron propagator [1]

By applying usual Feynman rules or making a Fourier transform, we obtain the momentum-space form for the proper fermion self-energy $-i\Sigma^f$. We have

$$-i\Sigma^f(p) = (e_0^f)^2 \int \frac{d^{\rm d}k}{(2\pi)^{\rm d}}(i\gamma_\mu)(iS^f(k))(iD^{\mu\nu}(p-k))(i\Gamma_\nu^f(k,p))\;. \tag{2.38}$$

Once we know the self energy $-i\Sigma^f$, we can determine the fermion propagator $S^f(p)$ by using

$$S^f(p) = \frac{1}{[(S_0^f)^{-1} - \Sigma^f(p)]} = \frac{1}{[p\!\!\!/ - m_0^f - \Sigma^f(p)]}, \tag{2.39}$$

which is obtained by multiplying Eq. (2.35) by $\left[S^f(x,y)\right]^{-1}$ and then going to momentum space.

2.4 Unrenormalised Equation for the Fermion Photon Vertex

This equation can be derived in a similar way to that used before, but the diagrammatic approach is more intuitive. In momentum space it reads [1]

$$i\Gamma^f_\mu(p',p) = i\gamma_\mu + \sum_g \int \frac{d^{\mathrm{d}}k}{(2\pi)^{\mathrm{d}}}(iS^g(q'))(i\Gamma^g_\mu(q',q))(iS^g(q))K^{gf}(q,q',k), \tag{2.40}$$

where $q' = q + k$, $q = p + k$ and K^{gf} is the fermion-antifermion scattering kernel with flavours g and f. The diagrammatic representation of this is given in Fig. (2.3). The amplitude M is 1-PI with respect to the fermion lines and does not contain

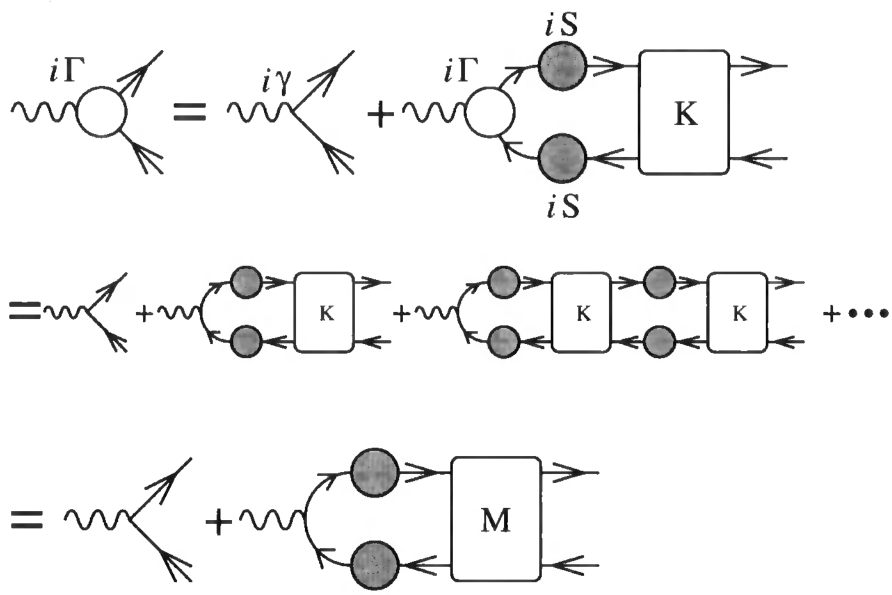

Figure 2.3: The Schwinger-Dyson equation for the proper vertex [1]

any fermion-antifermion annihilation contributions and as such does no contain any intermediate single photon state, since these would not be 1-PI contributions to $\Gamma^{f\mu}$ with respect to the photon line. The kernel K has been resummed to form M, which is given by

$$M = K + K(iS)^2K + K(iS)^2K(iS)^2K + \cdots = K + K(iS)^2M. \tag{2.41}$$

The matrix M satisfies its own equation involving n-point functions of higher order. In order to be tractable, we have to truncate the system of Schwinger-Dyson equations. This can be done by choosing an ansatz for M, i.e, for the kernel K. A common choice is the so-called *ladder approximation* which consists in approximating K by

$$K^{fg} = \delta^{fg}(e_0^f)^2(i\gamma_\mu)(iD_0^{\mu\nu})(i\gamma_\nu), \tag{2.42}$$

where D_0 is the bare photon propagator. We then iterate it to form M by replacing the full fermion propagator by its bare value. Another well-known approximation, is to replace the vertex by its bare value, i.e.

$$i\Gamma_\mu^f(p', p) = i\gamma_\mu, \tag{2.43}$$

which reduces the system to two-coupled equations. In the following section we will explore the case of the bare vertex approximation and its predictions for dynamical symmetry breaking.

2.5 The Bare Vertex Approximation

In the previous sections, we have derived the equations satisfied by the photon and fermion propagators and by the fermion-photon vertex. This system is not closed and is in fact quite complicated. The bare vertex approximation provides us with a simplification that permits the elimination of the equation for the vertex Γ^μ.

2.5.1 The equation for the fermion

In the bare vertex approximation, the equation for the self-energy Eq. (2.37), becomes

$$\Sigma^f(p) = ie_0^2 \int \frac{d^{\mathrm{d}}k}{(2\pi)^{\mathrm{d}}} \gamma_\mu S^f(k) D^{\mu\nu}(p-k)\gamma_\nu \,. \tag{2.44}$$

In order to solve this equation we first write the most general form for the fermion propagator. Because of its spinor structure, we have

$$S(p) = A(p^2)\not{p} + B(p^2), \tag{2.45}$$

where A and B are general functions of the Lorentz invariant p^2. In order to make èxplicit the appearance of a dynamical mass we rewrite this as

$$S(p) = \frac{\mathcal{F}(p^2)}{\not p - \Sigma(p^2)} = \frac{\mathcal{F}(p^2)\,(\not p - \Sigma(p^2))}{p^2 - \Sigma^2(p^2)}, \tag{2.46}$$

where $\mathcal{F}$ is called the *wave function renormalisation* and Σ is the *dynamical mass.* In order to find the equations satisfied by $\mathcal{F}$ and by Σ, we first take the trace of Eq. (2.44) and secondly multiply it by $\not p$ and then take the trace. We obtain

$$\begin{aligned} \frac{\Sigma(p^2)}{\mathcal{F}(p^2)} &= m_0 + \frac{1}{4}\mathrm{Tr}\left(\Sigma^f(p)\right), & (2.47)\\ \frac{1}{\mathcal{F}(p^2)} &= 1 - \frac{1}{4p^2}\mathrm{Tr}\left(\not p\Sigma^f(p)\right). & (2.48) \end{aligned}$$

We use the parametrisation of the fermion propagator in Eq. (2.46) to simplify the trace

$$\frac{\Sigma(p^2)}{\mathcal{F}(p^2)} = m_0 + \frac{ie^2}{4(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)}{k^2 - \Sigma^2(k^2)}\,\mathrm{Tr}\left[\gamma^\mu\left(\not k + \Sigma(k^2)\right)\gamma^\nu\right] D_{\nu\mu}(k-p), \tag{2.49}$$

$$\frac{1}{\mathcal{F}(p^2)} = 1 - \frac{ie_0^2}{4p^2(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)}{k^2 - \Sigma^2(k^2)}\,\mathrm{Tr}\left[\not p\,\gamma^\mu\left(\not k + \Sigma(k^2)\right)\gamma^\nu\right] D_{\nu\mu}(k-p). \tag{2.50}$$

In 4 dimensions the traces of a product of Dirac matrices can be computed using the following relations

$$\begin{aligned} \mathrm{Tr}\,[I] &= 4\,, \\ \mathrm{Tr}\,[\not k\,\not p] &= 4\,k.p\,, \qquad (2.51)\\ \mathrm{Tr}\,[\not k_1\not k_2\not k_3\not k_4] &= 4\left[(k_1.k_2)(k_3.k_4) - (k_1.k_3)(k_2.k_4) + (k_1.k_4)(k_2.k_3)\right], \\ \mathrm{Tr}\,[\not k_1,\ldots,\not k_n] &= 0 \quad , \qquad \text{if } n \text{ is odd}\,. \end{aligned}$$

We thus obtain the following relations

$$\frac{\Sigma(p^2)}{\mathcal{F}(p^2)} = m_0 + \frac{ie_0^2}{(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)\Sigma(k^2)}{k^2 - \Sigma^2(k^2)}\,\frac{g^{\mu\nu}}{q^2}\left[G(q^2)\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) - \xi_0\frac{q_\mu q_\nu}{q^2}\right], \tag{2.52}$$

$$
\begin{aligned}
\frac{1}{\mathcal{F}(p^2)} &= 1-\frac{ie_0^2}{p^2(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)}{k^2-\Sigma^2(k^2)}\left(p^\mu k^\nu+k^\mu p^\nu-k.p\,g^{\mu\nu}\right) && (2.53)\\
&\qquad\times\left\{-\frac{1}{q^2}\left[G(q^2)\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+\xi_0\frac{q_\mu q_\nu}{q^2}\right]\right\}.
\end{aligned}
$$

We now contract the indices μ and ν to write

$$
\begin{aligned}
\frac{\Sigma(p^2)}{\mathcal{F}(p^2)} &= m_0-\frac{ie_0^2}{(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)\Sigma(k^2)}{q^2(k^2-\Sigma^2(k^2))}\left\{3G(q^2)+\xi_0\right\}, && (2.54)\\
\frac{1}{\mathcal{F}(p^2)} &= 1+\frac{ie_0^2}{p^2(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)}{q^2(k^2-\Sigma^2(k^2))} && (2.55)\\
&\times\left\{G(q^2)\left[2\left(\frac{k^2p^2-(k.p)^2}{q^2}\right)-3k.p\right]+\xi_0\left[\frac{(k^2+p^2)\,k.p-2k^2p^2}{q^2}\right]\right\}.
\end{aligned}
$$

In order to proceed further, and be able to treat these equations numerically, we perform a Wick rotation on both the loop momentum k and the external momentum p. This Wick rotation is the usual procedure encountered in perturbative studies where it is well defined. Indeed, in the complex k_0 plane, the bare propagator for a fermion has a pole at k_0 given by

$$
k_0=\pm\sqrt{\mathbf{p}^2+m_0^2}\mp i\epsilon. \tag{2.56}
$$

By performing a Wick rotation, which amounts to rotating the real k_0 axis anticlockwise by an angle of $\pi/2$, one does not encounter any poles. In non-perturbative studies, we do not know *a priori*, the location of the singularities of the full-propagator and so it is an assumption that such a Wick rotation is well defined. It has been shown, in several studies [25] that in the bare vertex approximation and in others as well, the full fermion propagator has poles located in the complex k^2 plane. These singularities can be viewed from two different perspectives: they either are an artifact of the approximation or they are a genuine effect of the non-perturbative treatment and their presence signals the confinement of the fermion. In either case, it is first assumed that the propagators do not have such singularities, to be able to go to Euclidean space. These singularities thus cast doubt on the Wick rotation. If we maintain that the physics is based on Minkowski space, then we face a real problem of how to relate the two spaces, but if we assume that the fundamental metric is Euclidean, the singularities could be useful to signal dynamical symmetry breaking and confinement. In Lattice QCD, the metric is Euclidean and is able to predict physical quantities such as the masses of hadrons. We are here left with a real problem: *What is the fundamental metric?* This question could be addressed only when consistent calculations in Minkowski plane are undertaken, but for the moment it seems to be out of reach. Notwithstanding the theoretical issues with the Wick rotation, we perform it here to simplify the matter. We have

$$\frac{\Sigma(p^2)}{\mathcal{F}(p^2)} = m_0 + \frac{e_0^2}{(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)\Sigma(k^2)}{q^2(k^2+\Sigma^2(k^2))}\left\{3G(q^2)+\xi_0\right\}, \tag{2.57}$$

$$\frac{1}{\mathcal{F}(p^2)} = 1 - \frac{e_0^2}{p^2(2\pi)^4}\int d^4k\,\frac{\mathcal{F}(k^2)}{q^2(k^2+\Sigma^2(k^2))} \tag{2.58}$$
$$\times\left\{G(q^2)\left[2\left(\frac{k^2p^2-(k.p)^2}{q^2}\right)-3k.p\right]+\xi_0\left[\frac{(k^2+p^2)k.p-2k^2p^2}{q^2}\right]\right\}.$$

In Euclidean space a four-vector k_E has co-ordinates

$$k_E = (k_0, k_1, k_2, k_3), \tag{2.59}$$

which can be transformed to spherical coordinates by the following relations

$$\begin{aligned} k_0 &= k\cos\theta, \\ k_1 &= k\sin\theta\cos\phi, \\ k_2 &= k\sin\theta\sin\phi\cos\psi, \\ k_3 &= k\sin\theta\sin\phi\sin\psi\,, \end{aligned} \tag{2.60}$$

where $k = (k_0^2+k_1^2+k_2^2+k_3^2)^{1/2}$ is the modulus of the four-vector k_E. The integration ranges of the new variables are: $k \in [0,\infty]$, $\theta,\phi \in [0,\pi]$ and $\psi \in [0,2\pi]$.

If we now define the coupling constant $\alpha_0 \equiv e_0^2/4\pi$ and introduce the notation $x = p^2$, $y = k^2$ and $z = q^2$, then in spherical coordinates we are left with

$$\frac{\Sigma(x)}{\mathcal{F}(x)} = m_0 + \frac{\alpha}{2\pi^2}\int dy\,\frac{y\mathcal{F}(y)\Sigma(y)}{y+\Sigma^2(y)}\int d\theta\,\frac{\sin^2\theta}{z}\left\{3G(z)+\xi_0\right\}, \tag{2.61}$$

$$\frac{1}{\mathcal{F}(x)} = 1 - \frac{\alpha}{2\pi^2 x}\int dy\,\frac{y\mathcal{F}(y)}{y+\Sigma^2(y)}\int d\theta\,\frac{\sin^2\theta}{z} \tag{2.62}$$
$$\times\left\{G(z)\left[\frac{2xy\sin^2\theta}{z}-3\sqrt{yx}\cos\theta\right]+\xi_0\left[\frac{(y+x)\sqrt{yx}\cos\theta-2yx}{z}\right]\right\}.$$

Here, the angular integrals of the ξ_0-part can be computed analytically. We arrive then at the final form for the equations concerning the fermion propagator

$$\begin{aligned} \frac{\Sigma(x)}{\mathcal{F}(x)} &= m_0 + \frac{3\alpha}{2\pi^2}\int dy\, \frac{y\mathcal{F}(y)\Sigma(y)}{y+\Sigma^2(y)} \int d\theta\, \sin^2\theta\, \frac{G(z)}{z} \\ &+ \frac{\alpha\xi_0}{4\pi}\int dy\, \frac{\mathcal{F}(y)\Sigma(y)}{y+\Sigma^2(y)} \left[\frac{y}{x}\theta(x-y) + \theta(y-x)\right] \end{aligned} \tag{2.63}$$

$$\begin{aligned} \frac{1}{\mathcal{F}(x)} &= 1 - \frac{\alpha}{2\pi^2 x}\int dy\, \frac{y\mathcal{F}(y)}{y+\Sigma^2(y)} \int d\theta\, \sin^2\theta\, G(z) \left[\frac{2xy\sin^2\theta}{z^2} - \frac{3\sqrt{yx}\cos\theta}{z}\right] \\ &+ \frac{\alpha\xi_0}{4\pi}\int dy\, \frac{\mathcal{F}(y)}{y+\Sigma^2(y)} \left[\frac{y^2}{x^2}\theta(x-y) + \theta(y-x)\right] . \end{aligned} \tag{2.64}$$

2.5.2 The equation for the photon

To derive the equation for the photon propagator we recall that the polarisation tensor $\Pi_{\mu\nu}$ satisfies

$$i\Pi_{\mu\nu}(q) = (-1)\sum_f (e_0^f)^2 \int \frac{d^{\mathrm{d}}k}{(2\pi)^{\mathrm{d}}} \mathrm{Tr}[(i\gamma_\mu)(iS^f(k))(i\Gamma^{\mu f}(k,k+q))(iS^f(k+q))] \,, \tag{2.65}$$

with

$$\Pi_{\mu\nu}(q) = -q^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi(q^2). \tag{2.66}$$

The factor $\Pi(q^2)$ can be isolated by applying the projection operator $\mathcal{P}_{\mu\nu}$ defined as

$$\mathcal{P}_{\mu\nu} \equiv g^{\mu\nu} - n\frac{q^\mu q^\nu}{q^2}. \tag{2.67}$$

For any value of the integer value n, we have

$$\mathcal{P}_{\mu\nu}\Pi^{\mu\nu} = -3q^2\Pi(q^2). \tag{2.68}$$

This last relation permits us to write an equation for $\Pi(q^2)$, or introducing the function

$$G(q^2) = \frac{1}{1+\Pi(q^2)}, \tag{2.69}$$

we can write

$$\frac{1}{G(q^2)} = 1 - \frac{iN_f e_0^2 \mathcal{P}_{\mu\nu}}{3(2\pi)^4 q^2} \int d^4k \, \mathrm{Tr}\Big[\gamma^\mu \, S(k) \, \gamma^\nu \, S(p)\Big] \, . \tag{2.70}$$

After substituting the fermion propagator, we obtain

$$\frac{1}{G(q^2)} = 1 - \frac{iN_f e_0^2}{3(2\pi)^4 q^2} \int d^4k \, \frac{\mathcal{F}(k^2)\mathcal{F}(p^2)}{(k^2 - \Sigma^2(k^2))\,(p^2 - \Sigma^2(p^2))} \, \mathcal{P}_{\mu\nu} T^{\mu\nu}, \tag{2.71}$$

where $T^{\mu\nu}$ is given by

$$\begin{aligned} T^{\mu\nu} &= \mathrm{Tr}\Big[\gamma^\mu \,(\not k + \Sigma(k^2))\, \gamma^\nu \,(\not p + \Sigma(p^2))\Big] \\ &= 4\Big[k^\mu p^\nu + p^\mu k^\nu - \Big(k.p - \Sigma(k^2)\Sigma(p^2)\Big)\, g^{\mu\nu}\Big] \, . \end{aligned} \tag{2.72}$$

The product $\mathcal{P}_{\mu\nu} T^{\mu\nu}$ is readily computed as

$$\mathcal{P}_{\mu\nu} T^{\mu\nu} = 4\left[(n-2)k^2 - \frac{2n(k.q)^2}{q^2} + (n+2)k.q - (n-4)\Sigma(k^2)\Sigma(p^2)\right] \, . \tag{2.73}$$

So far we have not yet fixed the value of the integer n. In general such integrals are divergent and thus need regularisation. The most simple of the regularisation scheme is to use a cut-off Λ^2, which cuts off the momentum integration at $k^2 = \Lambda^2$. Ideally, we would hope that the introduction of the cut-off Λ^2 does not spoil the underlying gauge symmetry. If this statement is true when one uses dimensional regularisation, it is unfortunately not the case with a sharp cut-off regularisation. It can be seen at one loop already that the photon polarisation tensor does not factor as in Eq. (2.25) and it thus shows that in this scheme the photon acquire a mass. The presence of an ultra-violet cut-off Λ^2 introduces a mass term proportional to $g^{\mu\nu}$. In order to get rid of this term we choose $n = 4$ and obtain the so-called Brown-Pennington projector. With $n = 4$, we have $\mathcal{P}_{\mu\nu} g^{\mu\nu} = 0$ and we further assume that no mass term arises in the computation. After Wick rotation, we obtain the final form of the equation for $\Pi(q^2)$

$$\frac{1}{G(x)} = 1 + \frac{4N_f \alpha}{3\pi^2 x} \int dy \frac{y\mathcal{F}(y)}{y + \Sigma^2(y)} \int d\theta \, \sin^2\theta \, \frac{\mathcal{F}(z)}{z + \Sigma^2(z)} \Big[y(1 - 4\cos^2\theta) + 3\sqrt{yx}\cos\theta\Big] \, . \tag{2.74}$$

The three equations Eq. (2.63,2.64,2.74) are the basis of the study of dynamical symmetry breaking in QED_{3+1} in the bare vertex approximation. They were solved using a Chebyshev expansion for the functions Σ, $\mathcal{F}$ and G, as will be described in the next chapter. Even though we started with a theory defined in Minkowski space, we ended with equations defined in Euclidean space. We could have therefore started with an Euclidean theory and derived the same equations Eq. (2.63,2.64,2.74). We recall that the Euclidean metric is defined as

$$a \cdot b \;=\; \delta_{\mu\nu}\, a_\mu\, b_\nu = \sum_{i=1}^{4} a_i\, b_i \tag{2.75}$$

where $\delta_{\mu\nu}$ is the Kronecker delta. The Euclidean Dirac matrices are hermitian and satisfy the algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu}, \tag{2.76}$$

and the matrix γ^5 is given by

$$\gamma_5 = -\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4, \tag{2.77}$$

so that

$$\mathrm{Tr}[\gamma_5\,\gamma_\lambda\,\gamma_\mu\,\gamma_\nu\,\gamma_\rho] = -\,4\,\epsilon_{\lambda\mu\nu\rho}, \tag{2.78}$$

where $\epsilon_{\lambda\mu\nu\rho}$ is the completely antisymmetric Levi-Civita tensor in d = 4 dimensions. A possible representation of this algebra is

$$\gamma_4^E = \gamma^0 \;\text{ and }\; \gamma_j^E = -i\gamma^j, \;\; j = 1,2,3, \tag{2.79}$$

where γ^0 and γ^j can be taken to be any one of the commonly used Minkowski space representations of the usual Dirac algebra.

Once we have derived an expression in Minkowski space it can be transformed to obtain an equivalent in Euclidean space by using the following rules

$$\int^M d^4x^M \;\to\; -i\int^E d^4x^E, \tag{2.80}$$

$$\not\partial \;\to\; i\gamma^E \cdot \partial^E, \tag{2.81}$$

$$A_\mu\, B^\mu \;\to\; -A^E \cdot B^E\,. \tag{2.82}$$

In the following section we will give the results obtained in the bare vertex approximation.

2.5.3 Results

The equations Eq. (2.63,2.64,2.74) have been solved for a number of fermion species $N_f = 1$ and $N_f = 2$. Before quoting the results for the whole (Σ, $\mathcal{F}$, G) system, we can show further approximations that have been made to simplify this system.

One loop approximation

The momentum dependent coupling function $\tilde{\alpha}(x)$ defined in QED as

$$\tilde{\alpha}(x) = \alpha(\Lambda^2)G(x), \tag{2.83}$$

where $\alpha(\Lambda^2)$ is the bare coupling at the cut-off, can be approximated by its one-loop value $\alpha_{1\ell}$

$$\alpha_{1\ell}(x) = \frac{\alpha(\Lambda^2)}{1 + \frac{N_f \alpha(\Lambda^2)}{3\pi}\log(\frac{\Lambda^2}{x})}. \tag{2.84}$$

This rids us of one equation and in the Landau gauge ($\xi_0 = 0$), we are left with the system

$$\begin{aligned} \frac{\Sigma(x)}{\mathcal{F}(x)} &= \frac{3}{2\pi^2}\int dy\, \frac{y\mathcal{F}(y)\Sigma(y)}{y+\Sigma^2(y)} \int d\theta\, \sin^2\theta\, \frac{\alpha_{1\ell}(z)}{z}, \\ \frac{1}{\mathcal{F}(x)} &= 1 + \frac{1}{2\pi^2 x}\int dy\, \frac{y\mathcal{F}(y)}{y+\Sigma^2(y)} \\ &\quad \times \int d\theta\, \sin^2\theta\, \alpha_{1\ell}(z)\left(\frac{3\sqrt{xy}\cos\theta}{z} - \frac{2xy\sin^2\theta}{z^2},\right), \end{aligned} \tag{2.85}$$

where $z = x+y-2\sqrt{xy}\cos\theta$. The system can be either solved as it is written or with a further simplification by either approximating the wave function renormalisation $\mathcal{F}$ by 1, which is a good approximation in Landau gauge ($\xi_0 = 0$); or by using the angular approximation, implemented by writing

$$\alpha_{1\ell}(z) = \alpha_{1\ell}(x,y) = \alpha_{1\ell}(\max(x,y)). \tag{2.86}$$

The angular approximation leads to $\mathcal{F} = 1$ and permits us to compute the angular integrals analytically . There is then just one equation left and it involves only radial integration. It reads (in Landau gauge)

$$\Sigma(x) = \frac{3}{4\pi} \int dy \, \frac{y\Sigma(y)}{y + \Sigma^2(y)} \frac{\alpha_{1\ell}(\max(x,y))}{\max(x,y)}. \tag{2.87}$$

In this case there is dynamical symmetry breaking when the bare coupling $\alpha(\Lambda^2)$, is greater than a critical value α_c given by [3]

$$\begin{aligned} \alpha_c(N_f = 1) &= 1.99953\,, \\ \alpha_c(N_f = 2) &= 2.75233\,. \end{aligned} \tag{2.88}$$

If we adopt the approximation $\mathcal{F} = 1$, the equation to be solved is

$$\Sigma(x) = \frac{3\alpha(\Lambda^2)}{2\pi^2} \int dy \, \frac{y\Sigma(y)}{y + \Sigma^2(y)} \int d\theta \, \frac{\sin^2\theta}{z\left(1 + \frac{N_f\alpha(\Lambda^2)}{3\pi} \ln\frac{\Lambda^2}{z}\right)}\,. \tag{2.89}$$

The angular integration has to be performed numerically, which we handle easily using a Chebyshev expansion. The critical coupling in this case is [3]

$$\begin{aligned} \alpha_c(N_f = 1) &= 2.08431, \\ \alpha_c(N_f = 2) &= 2.99142, \end{aligned} \tag{2.90}$$

which are bigger than in the angular approximation.

The coupled system at one-loop approximation of Eq. (2.85) has also been solved and the critical couplings were the following [3]

$$\begin{aligned} \alpha_c(N_f = 1) &= 1.67280\,, \\ \alpha_c(N_f = 2) &= 2.02025. \end{aligned} \tag{2.91}$$

We can remark that in this case the critical couplings are smaller than in the former cases.

The system (Σ, $\mathcal{F}$, G)

This coupled system for the fermion and the photon has been solved using a Chebyshev expansion for the three functions and the critical couplings found were [3]

$$\alpha_c(\Lambda^2, N_f = 1) = 1.74102,$$
$$\alpha_c(\lambda^2, N_f = 2) = 2.22948, \tag{2.92}$$

which is an increase of 4% for $N_f = 1$ and of 10% for $N_f = 2$, when compared to the system $(\Sigma, \mathcal{F})$ at one-loop.

2.5.4 Improvement of the vertex

All the aforementioned values of the critical coupling assumed the bare vertex approximation. In order to improve the vertex, we need an ansatz that captures the relevant physics of the phenomenon. The underlying principles of the theory should be respected by the chosen ansatz. As has been mentioned before, the photon propagator satisfies a Ward-Takahashi identity (WTI), namely

$$q^\mu \Pi_{\mu\nu} = 0. \tag{2.93}$$

The vertex also obeys its own WTI, which relates the vertex to the fermion propagator. It reads

$$(k-p)_\mu \Gamma^\mu(k,p) = S^{-1}(k) - S^{-1}(p). \tag{2.94}$$

In the limit $(k-p) \to 0$, we obtain the Ward Identity

$$\Gamma^\mu(k,k) = \frac{\partial S^{-1}(k)}{\partial k^\mu}. \tag{2.95}$$

The structure of the WTI Eq. (2.94) shows that it only restricts the part longitudinal to the vector $(k-p)_\mu$. We will thus decompose the full vertex in two parts: one longitudinal and the other transverse to the vector $(k-p)_\mu$

$$\Gamma^\mu(k,p) = \Gamma^\mu_L(k,p) + \Gamma^\mu_T(k,p), \tag{2.96}$$

where the transverse part $\Gamma_T^\mu(k,p)$ has to be determined by other constraints such as having the right perturbative behaviour, being free of kinematical singularities or preserving multiplicative renormalisability.

Using the WTI Eq. (2.94) and Ward identity Eq. (2.95), Ball and Chiu [5] have determined the longitudinal part of the vertex, which is given by

$$\begin{aligned}\Gamma^\mu_{\mathrm{BC}}(k,p) = \Gamma_L^\mu(k,p) &= \frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}+\frac{1}{\mathcal{F}(p^2)}\right]\gamma^\mu \\ &+\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu(\not k+\not p)}{k^2-p^2} \\ &-\left[\frac{\Sigma(k^2)}{\mathcal{F}(k^2)}-\frac{\Sigma(p^2)}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu}{k^2-p^2}. \end{aligned} \tag{2.97}$$

In order to improve further the full vertex, one has to determine the structure of the transverse part. Its general structure can be expanded in a vector basis [5, 7, 8, 9]

$$\Gamma_T^\mu(k,p) = \sum_{i=1}^{8} \tau_i(k^2,p^2,q^2)\, T_i^\mu(k,p), \tag{2.98}$$

while assuring the transversality condition

$$(k-p)_\mu \Gamma_T^\mu(k,p) = 0\ , \qquad \text{and} \qquad \Gamma_T^\mu(p,p) = 0. \tag{2.99}$$

The vector basis $T_i^\mu(k,p)$ can be taken to be

$$\begin{aligned} T_1^\mu(k,p) &= p^\mu(k\cdot q) - k^\mu(p\cdot q)\,, \\ T_2^\mu(k,p) &= [p^\mu(k\cdot q) - k^\mu(p\cdot q)]\,(\not k+\not p)\,, \\ T_3^\mu(k,p) &= q^2\gamma^\mu - q^\mu \not q\,, \\ T_4^\mu(k,p) &= q^2\left[\gamma^\mu(\not p+\not k) - p^\mu - k^\mu\right] + 2(p-k)^\mu k^\lambda p^\nu \sigma_{\lambda\nu} \\ T_5^\mu(k,p) &= q_\nu \sigma^{\nu\mu}\,, \\ T_6^\mu(k,p) &= \gamma^\mu(k^2-p^2) - (k+p)^\mu(\not k - \not p)\,, \\ T_7^\mu(k,p) &= \frac{1}{2}(p^2-k^2)\left[\gamma^\mu(\not p+\not k) - p^\mu - k^\mu\right] + (k+p)^\mu k^\lambda p^\nu \sigma_{\lambda\nu}\,, \\ T_8^\mu(k,p) &= -\gamma^\mu k^\nu p^\lambda \sigma_{\nu\lambda} + k^\mu \not p - p^\mu \not k\,, \end{aligned} \tag{2.100}$$

with $q = k-p$ and $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. We thus have in general eight extra functions $\tau_i(k^2,p^2,q^2), i = 1,8$ to determine. By imposing multiplicative renormalisability, reproduction of perturbative results in the weak coupling limit and absence of free kinematical singularities in the massive case, Pennington and Curtis [6] were able to propose an ansatz for the transverse part of the vertex. They chose a simple form involving $T_6^\mu(k,p)$ only that meets these three requirements. The Curtis-Pennington (CP) vertex is defined as

$$\begin{aligned}\Gamma^\mu_{CP}(k,p) &= \Gamma^\mu_{BC}(k,p) \\ &+\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{(k^2+p^2)\left[\gamma^\mu(k^2-p^2)-(k+p)^\mu(\not k-\not p)\right]}{(k^2-p^2)^2+\left(\Sigma^2(k^2)+\Sigma^2(p^2)\right)^2}.\end{aligned} \tag{2.101}$$

Using this vertex, it is possible to write the SD equations for the fermion and photon propagators. They are [3]

$$\begin{aligned}\frac{\Sigma(x)}{\mathcal{F}(x)} &= m_0+\frac{\alpha}{2\pi^2}\int dy\,\frac{y\mathcal{F}(y)}{y+\Sigma^2(y)}\int d\theta\,\sin^2\theta\,\frac{G(z)}{z} \\ &\times\left\{3\Sigma(y)\left[A(y,x)+\tau_6(y,x)(y-x)\right]-\frac{1}{\mathcal{F}(x)}\left[\frac{\Sigma(y)-\Sigma(x)}{y-x}\right]\frac{2yx\sin^2\theta}{z}\right\} \\ &+\frac{\alpha\xi}{4\pi\mathcal{F}(x)}\int dy\,\frac{\mathcal{F}(y)}{y+\Sigma^2(y)}\left[\frac{y\Sigma(y)}{x}\theta(x-y)+\Sigma(x)\theta(y-x)\right],\end{aligned} \tag{2.102}$$

$$\begin{aligned}\frac{1}{\mathcal{F}(x)} &= 1-\frac{\alpha}{2x\pi^2}\int dy\,\frac{y\mathcal{F}(y)}{y+\Sigma^2(y)}\int d\theta\,\sin^2\theta\,\frac{G(z)}{z} \\ &\times\left\{A(y,x)\left[\frac{2yx\sin^2\theta}{z}-3\sqrt{yx}\cos\theta\right]\right. \\ &\left.+\left[B(y,x)(y+x)-C(y,x)\Sigma(y)\right]\frac{2yx\sin^2\theta}{z}-3\tau_6(y,x)(y-x)\sqrt{yx}\cos\theta\right\} \\ &-\frac{\alpha\xi}{4\pi\mathcal{F}(x)}\int dy\frac{\mathcal{F}(y)}{y+\Sigma^2(y)}\left[\frac{y\Sigma(y)\Sigma(x)}{x^2}\theta(x-y)-\theta(y-x)\right],\end{aligned} \tag{2.103}$$

$$\begin{aligned}\frac{1}{G(x)} &= 1+\frac{2N_f\alpha}{3\pi^2x}\int dy\,\frac{y\mathcal{F}(y)}{y+\Sigma^2(y)}\int d\theta\,\sin^2\theta\,\frac{\mathcal{F}(z)}{z+\Sigma^2(z)} \\ &\times\left\{2A(y,z)\left[y(1-y\cos^2\theta)+3\sqrt{yx}\cos\theta\right]\right. \\ &+B(y,z)\Big[\left(y+z-2\Sigma(y)\Sigma(z)\right)\left(2y(1-4\cos^2\theta)+3\sqrt{yx}\cos\theta\right) \\ &\qquad+3(y-z)\left(y-\Sigma(y)\Sigma(z)\right)\Big] \\ &-C(y,z)\left[\left(\Sigma(y)+\Sigma(z)\right)\left(2y(1-4\cos^2\theta)+3\sqrt{yx}\cos\theta\right)+3(y-z)\Sigma(y)\right] \\ &\left.-3\tau_6(y,z)(y-z)\left(y-\sqrt{yx}\cos\theta+\Sigma(y)\Sigma(z)\right)\right\},\end{aligned} \tag{2.104}$$

with

$$
\begin{aligned}
A(y,x) &= \frac{1}{2}\left[\frac{1}{\mathcal{F}(y)}+\frac{1}{\mathcal{F}(x)}\right], \\
B(y,x) &= \frac{1}{2(y-x)}\left[\frac{1}{\mathcal{F}(y)}-\frac{1}{\mathcal{F}(x)}\right], \\
C(y,x) &= -\frac{1}{y-x}\left[\frac{\Sigma(y)}{\mathcal{F}(y)}-\frac{\Sigma(x)}{\mathcal{F}(x)}\right], \\
\tau_6(y,x) &= \frac{y+x}{2\left[(y-x)^2+(\Sigma^2(y)+\Sigma^2(x))^2\right]}\left[\frac{1}{\mathcal{F}(y)}-\frac{1}{\mathcal{F}(x)}\right].
\end{aligned} \tag{2.105}
$$

These equations are much more complicated than the ones derived for the bare vertex and illustrate the fact that the construction of the vertex by using the WTI is a cumbersome approach. Moreover, we have seen that by using gauge symmetry principles we can improve the vertex (BC vertex). So far, we have only discussed bare equations. In order to make contact with the physical world, it would be profitable to define renormalised quantities and derive the equations they satisfy. In the next section we will show how the Schwinger-Dyson equations can be written for renormalised quantities and introduce a scheme, first used by Bloch [4] for the case of QCD, which respects multiplicative renormalisability without having to construct an explicit vertex.

2.6 Renormalised Equation in the MR Scheme

In this section, we introduce the concept of multiplicative renormalisability (MR) and illustrate its application to the case of QED. We then introduce a new truncation scheme called MR scheme that respects MR and derive the new SD equations in this scheme.

2.6.1 Multiplicative renormalisation

All the quantities, i.e. the fermion and photon propagators, we have thus far used are bare quantities. They were defined using the bare effective action $\Gamma[\bar{\psi},\psi,A_\mu]$ and they all potentially contain divergences, that have to be removed in order to make the theory meaningful. In perturbative studies, the expansion of $\Gamma[\bar{\psi}, \psi, A_\mu]$ in term of number of loops permits the calculation of the different couplings, which thus emerge as momentum dependent. The multiplicative renormalisation allows us to write quantities that are independent of the cut-off Λ^2, but instead depend on a mass scale μ^2, which corresponds to the scale at which, quantities are defined. For the fermion propagator, we have seen that it could be written

$$S^{-1}(p) = \frac{1}{\mathcal{F}(p^2)}\left[\not{p} - m(p^2)\right]. \tag{2.106}$$

The dependence on the cut-off is implicit and we should write

$$S^{-1}(p, \Lambda) = \frac{1}{\mathcal{F}(p^2, \Lambda^2)}\left[\not{p} - m(p^2, \Lambda^2)\right], \tag{2.107}$$

which makes the role of the cut-off Λ^2 more explicit and indicates the potential divergence of the function when Λ^2 is taken to infinity. The scale μ^2 can be defined as

$$S^{-1}(p) = \not{p} - m(\mu^2), \tag{2.108}$$

for $p^2 = \mu^2$ at a large value. This sets the wave function renormalisation $\mathcal{F}$ to unity at the same time. This approach is based on using bare fields $\bar{\psi}_0, \psi_0, A_0^\mu$, in the action and is thus called bare perturbative expansion. Alternatively, we can define multiplicatively renormalised fields and couplings in the following way

$$\psi_0^f = \sqrt{Z_2^f}\psi^f; \quad A_0^\mu = \sqrt{Z_3}A^\mu \ ; \quad e_0^f = \frac{Z_1^f}{Z_2^f\sqrt{Z_3}}e^f. \tag{2.109}$$

The infinities are absorbed in the fermion wave function renormalisation, Z_2, which relates the bare fermion field to the renormalised one; the photon wave function renormalisation, Z_3, which relates the bare photon field to the renormalised photon field; and the vertex renormalisation, Z_1, which relates the bare and renormalised charge. The WTI relating the vertex to the fermion propagator Eq. (2.94) permits us to write

$$Z_1^f = Z_2^f\,. \tag{2.110}$$

In order to obtain the renormalised action, we substitute in the bare action the bare fields by their expression in Eq. (2.109)

$$\begin{aligned} S^R_\xi[\bar{\psi},\psi,A_\mu] &= \int d^\mathrm{d}x \sum_{f=1}^{N_f} \left[Z_2^f \bar{\psi}^f \left(i\partial\!\!\!/ - m_0^f \right) \psi^f + Z_1^f e^f \bar{\psi}^f A\!\!\!/ \psi^f \right] \\ &- \int d^\mathrm{d}x \, \frac{Z_3}{4} F_{\mu\nu}F^{\mu\nu} - \frac{Z_3}{2\xi_0}(\partial_\mu A^\mu)^2. \end{aligned} \tag{2.111}$$

This action can be written in the usual form with the kinetic term normalised to unity for the fermion, by introducing so-called counter terms, which regulate the divergence encountered in the calculation of n-point functions. We then have

$$\begin{aligned} S^R_\xi[\bar{\psi},\psi,A_\mu] &= \int d^\mathrm{d}x \sum_{f=1}^{N_f} \bar{\psi}^f \left(i\partial\!\!\!/ - m^f + e^f A\!\!\!/ \right) \psi^f - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ & \int d^\mathrm{d}x \; + \sum_f \left[(Z_2^f - 1)\bar{\psi}^f i\partial\!\!\!/ \psi^f - \delta m^f \bar{\psi}^f \psi^f + \delta e^f \bar{\psi}^f A\!\!\!/ \psi^f \right] \\ &- \int d^\mathrm{d}x \, \frac{(Z_3 - 1)}{4} F_{\mu\nu}F^{\mu\nu}, \end{aligned} \tag{2.112}$$

with

$$\delta m^f = Z_2^f m_0^f - m^f, \tag{2.113}$$

$$\delta e^f = (Z_1^f - 1)e^f. \tag{2.114}$$

The renormalised gauge parameter ξ is

$$\xi_0 = Z_3 \xi, \tag{2.115}$$

and has no associated counter term. The WTI for the photon

$$q^\mu \Pi_{\mu\nu} = 0, \tag{2.116}$$

ensures that only the transverse part of the photon polarisation tensor gets corrected, i.e. the photon remains massless under renormalisation.

Using this renormalised action S^R_ξ, we can derive a renormalised effective action $\Gamma_R[\bar{\psi},\psi,A_\mu]$, which is the generating functional for renormalised 1PI graphs. The renormalised fermion and photon propagators will have the general form

$$\begin{aligned}(S_R^f)^{-1}(p) &= \not{p} - m^f - \Sigma_R^f(p), \\ D_R^{\mu\nu}(q) &= \frac{-g^{\mu\nu} + (q^\mu q^\nu/(q^2+i\epsilon))}{q^2+i\epsilon}\frac{1}{1+\Pi_R(q^2)} - \xi\frac{q^\mu q^\nu}{(q^2+i\epsilon)^2},\end{aligned} \tag{2.117}$$

subject to the renormalisation conditions

$$(S_R^f)^{-1}(p)\Big|_{\not{p}=m^f(p)} = \not{p} - m^f(p) \tag{2.118}$$

$$\Gamma_R^{f\mu}(p,p)\Big|_{\not{p}=m^f(p)} = \gamma^\mu \tag{2.119}$$

$$\Pi_R(0) = 0. \tag{2.120}$$

These renormalisation conditions form the so-called *on-shell* renormalisation and are the usual choice for QED. It ensures that the renormalised fermion propagator, $S_R^f(p)$, has a pole of residue one at the physical fermion mass $m^f(\mu)$ i.e.

$$\Sigma_R^f(p) = 0 \quad \text{at} \quad p^2 = \left[m^f(p)\right]^2; \tag{2.121}$$

that when an on-shell fermion of flavour f is probed with a zero-momentum photon we measure the physical charge, e^f and that an on-shell photon has a pole at $p^2 = 0$ with unit residue after vacuum polarisation corrections. This approach has been used for the study of perturbative QED and it has been very successful for the calculation of QED processes. Moreover, it has been shown that QED is perturbatively multiplicatively renormalisable and it is believed that this also holds non-perturbatively.

Using these definitions of the renormalised quantities, it is possible to write renormalised equations for the fermion and photon propagators and the vertex. Our goal is to write these equations in a scheme, where the explicit construction of the vertex is not needed. We will thus only need to know how to relate the renormalised fermion and photon propagators to their bare counterpart. We first write the most general expressions for the fermion and photon propagators in Euclidean space.

$$S(p) = \frac{Z(p^2,\Lambda^2)}{i\not{p} + M(p^2)}, \tag{2.122}$$

$$D_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{G(p^2,\Lambda^2)}{p^2} + \xi\frac{p^\mu p^\nu}{p^4}, \tag{2.123}$$

where we have explicitly written the dependence on Λ^2 and

$$G(p^2) = \frac{1}{1+\Pi(p^2)}. \tag{2.124}$$

The non-perturbative multiplicative renormalisation is achieved by writing

$$\begin{aligned} Z(p^2,\Lambda^2) &= Z_2(\mu^2,\Lambda^2) Z_R(p^2,\mu^2), & (2.125)\\ G(p^2,\Lambda^2) &= Z_3(\mu^2,\Lambda^2) G_R(p^2,\mu^2), & (2.126) \end{aligned}$$

which exchange the role of Λ^2 and μ^2. The functions Z_R and G_R are the renormalised dressing functions for the fermion and photon respectively and are normalised according to

$$Z_R(\mu^2,\mu^2) \; = \; G_R(\mu^2,\mu^2) \; = \; 1. \tag{2.127}$$

The running mass function $M(p^2)$ as defined in Eq. (2.122) is renormalisation point invariant, i.e. it does not depend on μ^2. The renormalised coupling is defined as

$$e(\mu^2) = \frac{Z_3^{1/2}(\mu^2,\Lambda^2) Z_2(\mu^2,\Lambda^2)}{Z_{1f}(\mu^2,\Lambda^2)} e_0(\Lambda^2) = Z_3^{1/2}(\mu^2,\Lambda^2) e_0(\Lambda^2), \tag{2.128}$$

where Z_{1f} is the vertex function renormalisation.

2.6.2 *MR* scheme

In the previous section, we have defined all the non-perturbative quantities needed to renormalise the SD equations for the fermion and photon propagators that we recall here for completeness. In Euclidean space, they are

$$[S_F(p)]^{-1} \;=\; [S_F^0(p)]^{-1} - e_0^2 \int \frac{d^4q}{(2\pi)^4} \gamma^\nu S_F(q) \Gamma^\nu(q,p) D^{\mu\nu}(r), \tag{2.129}$$

$$\begin{aligned} [D_{\mu\nu}(p)]^{-1} &= [D_{\mu\nu}^0(p)]^{-1} \\ &\quad -(-1) N_f e_0^2 \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S_F(q) \Gamma_\nu(q,-r) S_F(-r), & (2.130) \end{aligned}$$

with $r = (p-q)$.

MR equations for the fermion

Substituting the form of the fermion and photon propagators, we obtain the following equations

$$\frac{1}{Z(p^2,\Lambda^2)} = 1 + \frac{e_0^2(\Lambda^2)}{16\pi^4}\int d^4q\; Z(q^2,\Lambda^2)\frac{G(r^2,\Lambda^2)}{q^2+M(q^2)}U_Z(p^2,q^2,r^2,\Lambda^2), \tag{2.131}$$

$$\frac{M(p^2)}{Z(p^2,\Lambda^2)} = m_0(\Lambda^2) - \frac{e_0^2(\Lambda^2)}{16\pi^4}\int d^4qZ(q^2,\Lambda^2)M(q^2)\frac{G(r^2,\Lambda^2)}{q^2+M(q^2)}U_M(p^2,q^2,r^2,\Lambda^2), \tag{2.132}$$

where

$$\begin{aligned} U_Z(p^2,q^2,r^2,\Lambda^2) &= \frac{1}{4p^2r^2}\mathrm{Tr}\left[\not{p}\gamma_\mu\left(\not{q}+iM(q^2)\right)\Gamma_\nu(q,p,-r,\Lambda^2)\right] \\ &\times\left[\delta_\perp^{\mu\nu}(r)+\frac{\xi}{G(r^2,\Lambda^2)}\frac{r^\mu r^\nu}{r^2}\right], \qquad (2.133)\\ U_M(p^2,q^2,r^2,\Lambda^2) &= \frac{1}{4r^2}\mathrm{Tr}\left[\gamma_\mu\left(1-i\frac{\not{q}}{M(q^2)}\right)\right]\Gamma_\nu(p,q,-r,\Lambda^2) \\ &\times\left[\delta_\perp^{\mu\nu}(r)+\frac{\xi}{G(r^2,\Lambda^2)}\frac{r^\mu r^\nu}{r^2}\right], \qquad (2.134)\end{aligned}$$

with $\delta_\perp^{\mu\nu}(r)=\delta^{\mu\nu}-r^\mu r^\nu/r^2$. The two functions U_M and U_Z are the kernels for the integral equations. They implicitly depend on Z and M, through their dependence on the full vertex Γ^μ, using the WTI identity. We have made explicit the presence of the cut-off Λ^2, which regulates the theory in a way that the bare mass m_0 is cut-off dependent, ie. $m_0 = m_0(\Lambda^2)$, but the dynamical mass M is regulator independent $M = M(p^2)$.

The relations defining the renormalised dressed functions of Eq. (2.125) and the definition of the renormalised coupling in Eq. (2.128) , allows us to derive equations involving the functions $Z_R(p^2,\mu^2)$ and $\alpha(\mu^2)=e^2(\mu^2)/4\pi$. We have

$$\begin{aligned}\frac{1}{Z_R(p^2,\mu^2)} &= Z_2(\mu^2,\Lambda^2)+\frac{\alpha(\mu^2)}{4\pi^3}Z_2^2(\mu^2,\Lambda^2) \qquad (2.135)\\ &\times\int d^4q\; Z_R(q^2,\mu^2)\frac{G_R(r^2,\mu^2)}{q^2+M(q^2)}U_Z(p^2,q^2,r^2,\Lambda^2),\end{aligned}$$

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2) - \frac{\alpha(\mu^2)}{4\pi^3}Z_2^2(\mu^2,\Lambda^2) \times \int d^4q Z_R(q^2,\mu^2)M(q^2)\frac{G_R(r^2,\mu^2)}{q^2+M(q^2)}U_M(p^2,q^2,r^2,\Lambda^2). \tag{2.136}$$

In these equations, it is usual to use the WTI for the vertex to write $Z_2(\mu^2,\Lambda^2) = Z_{1f}(\mu^2,\Lambda^2)$ and to include a factor Z_{1f} in the integral so as to renormalise the fermion vertex Γ^μ. Here we will not proceed in this way and leave for the moment the vertex unrenormalised. The Λ^2 dependence of the integrals is such that it is cancelled by the seeds $m_0(\Lambda^2)$ and $Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2)$ to produce renormalised quantities Z_R and M. The factor $Z_2(\mu^2,\Lambda^2)$ is replaced in these two equations by its definition

$$Z_2(\mu^2,\Lambda^2) = \frac{Z(p^2,\Lambda^2)}{Z_R(p^2,\mu^2)}, \tag{2.137}$$

to produce the following equations

$$\frac{1}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2) + \frac{\alpha(\mu^2)}{4\pi^3} \times \int d^4q \frac{1}{Z_R(q^2,\mu^2)}\frac{G_R(r^2,\mu^2)}{q^2+M(q^2)}Z^2(q^2,\Lambda^2)U_Z(p^2,q^2,r^2,\Lambda^2), \tag{2.138}$$

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2) - \frac{\alpha(\mu^2)}{4\pi^3} \times \int d^4q \frac{M(q^2)}{Z_R(q^2,\mu^2)}\frac{G_R(r^2,\mu^2)}{q^2+M(q^2)}Z^2(q^2,\Lambda^2)U_M(p^2,q^2,r^2,\Lambda^2). \tag{2.139}$$

In order to derive a system of equations relating the fermion propagator to the QED coupling $\alpha(p^2)$, we note that the product $\alpha(\mu^2)G_R(r^2,\mu^2)$ does not depend on the renormalisation point μ^2 and can be expressed in terms of bare quantities only

$$\alpha(\mu^2)G_R(r^2,\mu^2) = \alpha(\Lambda^2)G(r^2,\Lambda^2), \tag{2.140}$$

which allows us to choose the value of μ^2 at our convenience. Choosing $\mu^2 = r^2$, in the integral equation and noting that $G_R(r^2,r^2) = 1$, we obtain

$$\frac{1}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2) + \frac{1}{4\pi^3}\int d^4q \frac{1}{Z_R(q^2,\mu^2)}\frac{\alpha(r^2)}{q^2+M(q^2)}Z^2(q^2,\Lambda^2)U_Z(\Lambda^2), \tag{2.141}$$

$$
\begin{aligned}
\frac{M(p^2)}{Z_R(p^2,\mu^2)} &= Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2) \\
&\quad -\frac{1}{4\pi^3}\int d^4q \frac{M(q^2)}{Z_R(q^2,\mu^2)}\frac{\alpha(r^2)}{q^2+M(q^2)}Z^2(q^2,\Lambda^2)U_M(\Lambda^2),
\end{aligned}
\tag{2.142}
$$

where the dependence of the kernels U_M and U_Z on the momenta p, q, r is not written. So far we have made no approximations, but now introduce the truncation that preserves Multiplicative Renormalisability in a non-perturbative way. We note that in both equations the full unrenormalised vertex Γ^μ is multiplied by the unrenormalised dressing function $Z(q^2,\Lambda^2)$. The truncation consists of assuming that the factor $Z^2(q^2,\Lambda^2)$ cancels both the nonperturbative correction to the bare fermion-photon vertex and the corrections due to the integration. This scheme was first proposed by J.Bloch [4] for the case of QCD. In this case the truncation consists of assuming that a factor Z^2/G^2, where G is the ghost dressing function cancels the nonperturbative correction to the bare quark-gluon vertex and the corrections due to the integration. This can be supported by recalling the WTI for the quark

$$
(p-k)_\mu \Gamma^\mu_{qg}(p,k,p-k) = G[(p-k)^2][S^{-1}(k) - S^{-1}(k)], \tag{2.143}
$$

which shows that the full quark-gluon vertex receives G/Z non-perturbative corrections. However, it has been shown by Mandelstam [10], that perturbative loop corrections to the propagator introduce a double factor G/Z. The cancellation mechanism is thus plausible and it is assumed that it is a non-perturbative phenomenon. In QED, the WTI is

$$
(p-k)_\mu \Gamma^\mu(p,k) = [S^{-1}(k) - S^{-1}(k)] \tag{2.144}
$$

and here as well we assume that a double $1/Z$ corrections appears that is cancelled by the factor Z^2 in the integrals. This *Ansatz* is consistent with perturbation theory at weak coupling and thus yields the correct resumed perturbative behaviour of the electron propagator. We will thus make the replacement

$$Z^2(q^2,\Lambda^2)U_Z(p^2,q^2,r^2,\Lambda^2) \rightarrow U_Z^0(p^2,q^2,r^2), \quad (2.145)$$

$$Z^2(q^2,\Lambda^2)U_M(p^2,q^2,r^2,\Lambda^2) \rightarrow U_M^0(p^2,q^2,r^2), \quad (2.146)$$

where U_Z^0 and U_M^0 are calculated by replacing the full vertex $\Gamma^\mu(p,q)$ by its bare expression $i\gamma^\mu$. The equations for the fermion propagator are thus

$$\frac{1}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2) + \frac{1}{4\pi^3}\int d^4q\, \frac{1}{Z_R(q^2,\mu^2)}\frac{\alpha(r^2)}{q^2+M(q^2)}U_Z^0, \quad (2.147)$$

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2) - \frac{1}{4\pi^3}\int d^4q \frac{M(q^2)}{Z_R(q^2,\mu^2)}\frac{\alpha(r^2)}{q^2+M(q^2)}U_M^0. \quad (2.148)$$

The calculation U_Z^0 and U_M^0 is straightforward, as it is the same as the one that was performed for the bare vertex approximation. In Landau gauge where $\xi = 0$, we have (see Eq. (2.54,2.55))

$$U_Z^0(p^2,q^2,r^2) = \frac{1}{p^2r^2}\left[3p.q - \frac{2\left[p^2q^2-(p.q)^2\right]}{r^2}\right], \quad (2.149)$$

$$U_Z^0(p^2,q^2,r^2) = -\frac{3}{r^2}. \quad (2.150)$$

The main interest of this truncation scheme is that it preserves multiplicative renormalisability and that there is no need to construct a fermion-photon vertex to satisfy this property. Multiplicative renormalisability is preserved if solutions of the equations renormalised at the scales μ^2 and ν^2 are related by

$$Z_R(p^2,\nu^2) = \frac{Z_R(p^2,\mu^2)}{Z_R(\nu^2,\mu^2)},$$

$$Z_2(\nu^2,\Lambda^2) = Z_R(\nu^2,\mu^2)Z_2(\mu^2,\Lambda^2). \quad (2.151)$$

If we assume that we have found a solution $(Z_R(p^2,\mu^2), M(p^2))$ of the equations Eq. (2.141,2.142), for all p^2 and renormalised at the scale μ^2, then by multiplying these two equations by the function $Z_R(\nu^2,\mu^2)$, it is clear that $Z_R(p^2,\nu^2)$ and the original $M(p^2)$ are solutions of the system with the renormalisation condition

$Z_R(\nu^2, \nu^2) = 1$. We can also note that this truncation satisfies the leading order re-summed perturbation theory, as was demonstrated in [12, 11]. Our present scheme does not display the same Z dependence but because the UV limits are identical, i.e. $Z_R(p^2) = 1$ to leading order, then the mass functions have the same UV behaviour.

MR equation for the photon

The two equations relating the dressing function Z and the mass function M have been written to involve the coupling function $\alpha(p^2)$, that satisfies its own integral equation. Starting from the SD equation for the photon propagator, we can derive the equation for the coupling in the MR scheme. It is easier to remember that the MR scheme can be obtained from the bare vertex approximation by modifying the Z dependence, i.e. absorbing a factor Z^2 in the full vertex. From the bare vertex approximation equation for the photon propagator Eq. (2.74), we can derive the equation in the MR scheme. In Landau gauge, it reads as

$$\begin{aligned}\frac{1}{\alpha(x)} &= \frac{4\pi Z_3(\mu^2,\Lambda^2)}{e^2(\mu^2)} + \frac{4N_f}{3\pi^2 x}\int dy \frac{y}{y+\Sigma^2(y)} \\ &\times \int d\theta \, \sin^2\theta \, \frac{1}{z+\Sigma^2(z)} \left[y(1-4\cos^2\theta) + 3\sqrt{yx}\cos\theta\right] ,\end{aligned} \tag{2.152}$$

with $x = q^2$, $y = k^2$ and $z = p^2 = (k-q)^2$ and

$$\alpha(q^2) = \frac{e^2(\mu^2) G_R(q^2,\mu^2)}{4\pi}. \tag{2.153}$$

Recapitulation

For completeness, we write the QED equation in the MR scheme in the Landau gauge.

$$\begin{aligned}\frac{1}{Z_R(x,\mu^2)} &= Z_2(\mu^2,\Lambda^2) + \frac{1}{2\pi^2}\int dy \, \frac{1}{Z_R(y,\mu^2)} \frac{y}{y+M^2(y)} \\ &\times \int_0^\pi d\theta \sin^2\theta \frac{\alpha(z)}{x}\left(\frac{3\sqrt{xy}\cos\theta}{z} - \frac{2xy\sin^2\theta}{z^2}\right),\end{aligned} \tag{2.154}$$

$$\frac{M(x)}{Z_R(x,\mu^2)} = Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2) + \frac{3}{2\pi^2}\int dy \frac{M(y)}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)} \times \int_0^\pi d\theta \sin^2\theta \frac{\alpha(z)}{z}, \tag{2.155}$$

$$\frac{1}{\alpha(x)} = \frac{4\pi Z_3(\mu^2,\Lambda^2)}{e^2(\mu^2)} + \frac{4N_f}{3\pi^2 x}\int dy \frac{y}{y+\Sigma^2(y)} \int d\theta \, \sin^2\theta \frac{1}{z+\Sigma^2(z)}\left[y(1-4\cos^2\theta) + 3\sqrt{yx}\cos\theta\right]. \tag{2.156}$$

These equations involve the wave function renormalisation Z_2 and Z_3, which will be removed by subtracting each equation by the same equation at some particular momentum σ^2, which can be chosen arbitrarily. In the next chapter, we will introduce the numerical method that was employed to solve this system in different approximations, i.e. quenched QED, when the coupling is replaced by its bare value, one loop approximation and then the full system.

Chapter 3

Numerical Method For Integral Equations

In this chapter, we introduce a powerful numerical method to solve the non-linear integral equations for the QED system. The system of equations Eq. (2.154-2.156), involve the coupling function $\alpha(q^2)$ in the angular integral, which is the term that proves to create difficulties in the numerical treatment of the system. Had it not been for the presence of such a term, we could have used a traditional approach, which consists of replacing the unknown functions $Z(x)$, $M(x)$ and $\alpha(x)$ by their values $Z(x_i)$, $M(x_i)$ and $\alpha(x_i)$ at some grid-points $x_i, i = 1, N$. This is equivalent to a linear fit of the unknown functions between the points x_i. The reconstructed functions are continuous but not smooth and this leads to numerical problems, namely there is not proper cancellation of the divergences and the coupling function $\alpha(x)$ has a non-physical behaviour [3]. In order to avoid such problems, we use an approximation that maintains continuity and smoothness of the functions. This is achieved by making a Chebyshev expansion. We thus transform a system of integral equations into a non-linear system of (almost) algebraic equations, where the variables are the coefficients of the Chebyshev expansion. This non-linear system can then be solved by the Newton method, or better by the so-called modified Newton method, which is a globally convergent method. Once we have determined the expansion coefficients, it is easy to reconstruct the functions.

3.1 Chebyshev Expansion

3.1.1 Chebyshev polynomials

The Chebyshev polynomial of degree n is denoted $T_n(x)$. It has an explicit representation given by

$$T_n(x) = \cos(n \arccos(x)). \tag{3.1}$$

Its argument x has thus the range $-1 \leq x \leq 1$. Even though, the definition of T_n involves a trigonometric function, it is nevertheless a polynomial in x and an explicit expression for T_n can be calculated easily, analytically for the first few terms or by using a recurrence relation, as follows [13]

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 4x^4 - 8x^2 + 1 \\
&\dots \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \qquad n \geq 1.
\end{aligned} \tag{3.2}$$

We plot the first few Chebyshev polynomials $T_n(x)$, $n = 0,\dots,4$ in Fig. (3.1). The Chebyshev polynomial $T_n(x)$ has n zeros located at the points

$$x_k^z = \cos\left(\frac{\pi\left(k - \frac{1}{2}\right)}{n}\right) \qquad k = 1, 2, \dots, n. \tag{3.3}$$

It also has $n+1$ extrema, both minima and maxima, in the interval $[-1, 1]$, located at

$$x_k^e = \cos\left(\frac{\pi k}{n}\right) \qquad k = 0, 1, \dots, n. \tag{3.4}$$

The maxima always occur at 1 and the minima always at -1. The Chebyshev polynomials also satisfy two orthogonality relations in the interval $[-1, 1]$, one continuous, over a weight $(1 - x^2)^{1/2}$, which involves an integral over the argument x

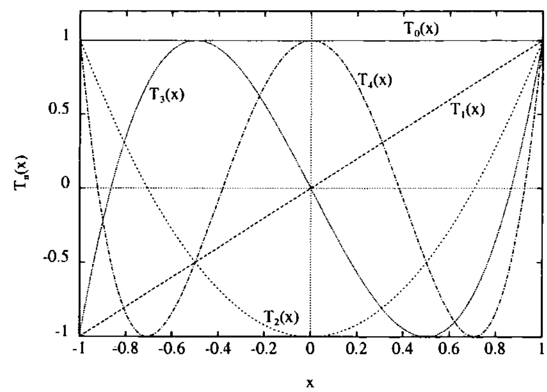

Figure 3.1: Chebyshev Polynomials $T_n(x)$ for n=0,. . . ,4.

and one discrete, which involves a sum over the index k of the zeros $x_k, k = 1, \ldots, m$ of $T_m(x)$. They are

$$\int_{-1}^{1} dx \frac{1}{\sqrt{1-x^2}} T_i(x)T_j(x) = \begin{cases} 0 & i \neq j \\ \pi/2 & i = j \neq 0 \\ \pi & i = j = 0 \end{cases} \tag{3.5}$$

$$\sum_{k=1}^{m} T_i(x_k)T_j(x_k) = \begin{cases} 0 & i \neq j \\ m/2 & i = j \neq 0 \\ m & i = j = 0. \end{cases} \tag{3.6}$$

3.1.2 Chebyshev approximation

Because of the continuous orthogonality relation Eq. (3.5), the Chebyshev polynomials can be used as a basis for any continuous function. Any function $f(x)$, with argument $-1 \leq x \leq 1$, can thus be rewritten in terms of Chebyshev polynomials as follows

$$f(x) \equiv \sum_{j=1}^{\infty} c_j T_{j-1}(x) - \frac{c_1}{2} = \sum_{j=1}^{\infty}{}' c_j T_{j-1}(x) \tag{3.7}$$

where the c_j are an infinite number of expansion coefficients and where the shorthand $\sum'$ is defined. We will make an approximation to the function $f(x)$, by truncating the sum to $j \leq N$

$$f(x) \approx \sum_{j=1}^{N}{}' c_j T_{j-1}(x) \equiv \sum_{j=1}^{N} c_j T_{j-1}(x) - \frac{c_1}{2}, \tag{3.8}$$

such that the approximation becomes *exact* at the N zeros of $T_N(x)$.

For the zeros x_k, $(k = 1, N)$, we have

$$f(x_k) = \sum_{j=1}^{N}{}' c_j T_{j-1}(x_k), \qquad k = 1, \ldots, N. \tag{3.9}$$

By multiplying both sides by $T_i(x_k)$, with $i < N$ and summing over the index k, we obtain

$$\sum_{k=1}^{N} T_i(x_k) f(x_k) = \sum_{j=1}^{N}{}' c_j \sum_{k=1}^{N} T_i(x_k) T_{j-1}(x_k). \tag{3.10}$$

We now use the discrete orthogonality relation (3.6), which yields

$$\sum_{k=1}^{N} T_i(x_k) f(x_k) = \frac{N}{2} c_{i+1}, \tag{3.11}$$

so the expansion coefficients c_j of Eq. (3.8) can be written as

$$c_j = \frac{2}{N} \sum_{k=1}^{N} T_{j-1}(x_k) f(x_k). \tag{3.12}$$

If we substitute the expression Eq. (3.3) for the zeros of $T_N(x)$ we have

$$c_j = \frac{2}{N} \sum_{k=1}^{N} T_{j-1} \left[\cos\left(\frac{(k-1/2)\pi}{N} \right) \right] f \left[\cos\left(\frac{(k-1/2)\pi}{N} \right) \right]. \tag{3.13}$$

We use the explicit representation of the Chebyshev polynomial $T_j(x)$ in Eq. (3.1) to obtain an explicit representation for the expansion coefficients c_j

$$c_j = \frac{2}{N} \sum_{k=1}^{N} \cos\left(\frac{(j-1)(k-1/2)\pi}{N} \right) f \left[\cos\left(\frac{(k-1/2)\pi}{N} \right) \right], \qquad j = 1, \ldots, N. \tag{3.14}$$

The use of Chebyshev polynomials for the expansion is not a pedantic one. There are other polynomials that could be used. However, the best choice seems to be the Chebyshev polynomials. If we make a different approximation using another set of polynomials truncated at order N, then by choosing an integer m, such that $m < N$, then the approximation

$$f(x) \approx \sum_{j=1}^{m}{}' c_j T_{j-1}(x), \tag{3.15}$$

yields the "most accurate" approximation of degree m [13]. Because the T_j are all bounded between ± 1, the difference Δf between a truncation at order N, and a truncation at order m is smaller than the sum of the $c_k, k = m+1, \ldots, N$

$$\Delta f = \sum_{j=1}^{N}{}' c_j T_{j-1}(x) - \sum_{j=1}^{m}{}' c_j T_{j-1}(x) \leq \sum_{k=m+1}^{n} c_k. \qquad (3.16)$$

If all the c_k, are rapidly decreasing, then Δf is dominated by the term $c_{m+1}T_m(x)$, which is an oscillatory function with $m+1$ extrema smoothly distributed over the interval [-1,1]. The error generated by replacing the function by its expansion is thus smeared out over the complete interval. Moreover, it can be shown [13], that the Chebyshev polynomial is very close to the so-called *minimax polynomial*, which has the smallest maximum deviation from the true function. The advantage of the Chebyshev polynomial is that it is easily computed unlike the minimax polynomial.

3.1.3 Summation of Chebyshev polynomials

Once we are able to compute the expansion coefficients c_j in Eq. (3.14), we will need to evaluate the function $f(x)$. One approach is to use the recurrence relation Eq. (3.2) to generate the different $T_j, j = 1, N-1$ and then sum the calculated polynomials weighted by their associated coefficient. This approach is numerically dangerous, because the recurrence relation Eq. (3.14) is unstable and yields exponentially growing solutions. Fortunately, there is a way to generate the values of the $T_j(x)$'s, and accumulate the sum to form $f(x)$ in one go. *Clenshaw's recurrence formula* is an elegant way to evaluate a sum of coefficients times functions that obey a recurrence formula, which is what we have when we expand a function over the basis of the Chebyshev polynomials. We thus want to evaluate

$$f(x) = \sum_{k=0}^{N} c_k T_k(x), \qquad (3.17)$$

where the functions $T_k(x)$ obey the recurrence relation

$$T_{n+1}(x) = \alpha(n,x)T_n(x) + \beta(n,x)T_{n-1}(x), \qquad (3.18)$$

with some functions $\alpha(n,x)$ and $\beta(n,x)$. First define the quantities y_k, by the following decreasing recurrence

$$y_{N+2} = y_{N+1} = 0\,, \qquad (3.19)$$

$$y_k = \alpha(n,x)y_{k+1} + \beta(n,x)y_{k+2} + c_k\,, \qquad (k = N, N-1, \ldots, 1)\,. \quad (3.20)$$

If we solve Eq. (3.19) for c_k on the left, and then make explicitly the sum Eq. (3.17), we will get the following pattern [13]

$$\begin{aligned} f(x) = \; & \ldots \\ & + [y_8 - \alpha(8,x)y_9 - \beta(9,x)y_{10}]\,T_8(x) \\ & + [y_7 - \alpha(7,x)y_8 - \beta(8,x)y_9]\,T_7(x) \\ & + [y_6 - \alpha(6,x)y_7 - \beta(7,x)y_8]\,T_6(x) \\ & + [y_5 - \alpha(5,x)y_6 - \beta(6,x)y_7]\,T_5(x) \\ & + \ldots \\ & + [y_2 - \alpha(2,x)y_3 - \beta(3,x)y_4]\,T_2(x) \\ & + [y_1 - \alpha(1,x)y_2 - \beta(2,x)y_3]\,T_1(x) \\ & + [c_1 + \beta(1,x)y_2 - \beta(1,x)y_2]\,T_0(x), \end{aligned} \quad (3.21)$$

where we have added and subtracted $\beta(1,x)y_2$ in the last line. For example, the terms containing a factor y_8 in Eq. (3.21) sum to zero because of the recurrence relation Eq. (3.18), and this applies to all the other y_k. The only term left is thus the sum we wish to evaluate. We have

$$f(x) = \beta(1,x)T_0(x)y_2 + T_1(x)y_1 + T_0(x)c_1. \quad (3.22)$$

The two equations Eq. (3.19) and Eq. (3.22) are *Clenshaw's recurrence formula* for evaluating the sum of coefficients times functions that obey a recurrence formula. If we apply Clenshaw's formula to the Chebyshev polynomials obeying the recurrence relation Eq. (3.2), the function approximating $f(x)$ in Eq. (3.8) is given by

$$\begin{aligned} y_{N+2} &= y_{N+1} = 0\,, \\ y_j &= 2xy_{j+1} - y_{j+2} + c_j \qquad j = N, N-1, \ldots, 2\,, \\ f(x) &\equiv y_0 = xy_2 - y_3 + \frac{1}{2}c_1. \end{aligned} \quad (3.23)$$

So far, we assumed that the function has an argument x in the same range as the Chebyshev polynomial. If the range of the argument is more general, i.e. $b \leq x \leq a$, we only make a change of variables to map x to the range $[-1, 1]$. We define the variable s as

$$s = \frac{x - \frac{1}{2}(b+a)}{\frac{1}{2}(b-a)}, \tag{3.24}$$

which satisfies,

$$x \in [a, b] \mapsto s \in [-1, 1]. \tag{3.25}$$

The Chebyshev approximation is now

$$f(x) \approx \sum_{j=1}^{N}{}' c_j T_{j-1}(s), \tag{3.26}$$

where x is mapped into s using Eq. (3.24).

3.2 Globally Convergent Method for Non-Linear Systems

As we have already mentioned in the introduction to the chapter, the system of integral equations is rewritten as a system of (almost) non-linear algebraic equations, which is equivalent to finding the zero of a vector function $\mathbf{F}(\mathbf{x})$, where $\mathbf{x}$ is the vector representing the expansion coefficients. We first start by discussing the Newton method in one dimension and then generalise to several dimensions.

3.2.1 Newton method

Suppose we have an estimate x, of the zero x_z of a function f in one dimension

$$f(x_z) = 0. \tag{3.27}$$

Algebraically, the Newton method consists of writing a Taylor expansion for $x_z = x + \delta$, where δ is small and x is an estimate of x_z. We have

$$f(x+\delta) = f(x) + \delta f'(x) + \delta^2 \frac{f''(x)}{2} + \ldots \tag{3.28}$$

because $f(x_z) = f(x+\delta) = 0$, it implies

$$\delta = -\frac{f(x)}{f'(x)} \tag{3.29}$$

The iteration procedure will thus be

$$x_{i+1} = x_i + \delta = x_i - \frac{f(x_i)}{f'(x_i)}, \tag{3.30}$$

if of course the derivative at the point x_i does not vanish.

The advantage of this method is that it yields a fast rate of convergence, once we are close to the zero. Indeed we have

$$\begin{aligned} f(x+\epsilon) &= f(x) + \epsilon f'(x) + \epsilon^2 \frac{f''(x)}{2} + \dots, \\ f'(x+\epsilon) &= f'(x) + \epsilon f''(x) + \dots. \end{aligned} \tag{3.31}$$

By applying the Newton formula Eq. (3.30), we get

$$\begin{aligned} \epsilon_{i+1} &= \epsilon_i - \frac{f(x_i)}{f'(x_i)}, \\ &= -\epsilon_i^2 \frac{f''(x)}{f'(x)}, \end{aligned} \tag{3.32}$$

where x is the zero of f. The Newton method thus converges quadratically when we are near a root.

The case of several dimensions is similar to the previous one. We would have N functions (F_i, $i = 1, N$), involving N variables (x_i, $i = 1, N$), to be zeroed

$$F_i(x_1, x_2, \dots, x_N) = 0 \qquad i = 1, 2, \dots, N. \tag{3.33}$$

If $\mathbf{x}$ denotes the vector $(x_1, x_2, \dots, x_N)$, we can write the expansion

$$F_i(\mathbf{x} + \delta\mathbf{x}) = F_i(\mathbf{x}) + \sum_{j=1}^{N} J_{ij}\delta x_j + \mathcal{O}(\delta\mathbf{x}^2), \tag{3.34}$$

where J_{ij} is *the Jacobian matrix* associated to $\mathbf{F}$

$$J_{ij} = \frac{\partial F_i}{\partial x_j}. \tag{3.35}$$

The increment $\delta\mathbf{x}$ satisfies the equation

$$\mathbf{J}.\delta\mathbf{x} = -\mathbf{F}. \tag{3.36}$$

This equation is a linear system that can be solved using the standard methods. It yields the increment of the Newton method

$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}} + \delta\mathbf{x}, \tag{3.37}$$

$$\delta\mathbf{x} = -\mathbf{J}^{-1}.\mathbf{F} \tag{3.38}$$

3.2.2 Global method

As we have seen, the Newton Method has a quadratic rate of convergence once we are close to the root. Unfortunately, if we start far from the root, convergence is not assured. Simple examples where the convergence fails exist [13] and it is thus necessary to find a method for which convergence occurs for almost any initial guess. A global method is one that will converge to a solution for almost any starting point. We would like to employ a method that ensures both quadratic convergence à la Newton plus global convergence to the root after each iteration [13].

After each Newton iteration Eq. (3.37), we should ask ourselves if we really want to keep the new approximation vector $\mathbf{x}_{\text{new}}$ or not. From the vector function $\mathbf{F}$, we can form its normalised squared norm f defined as

$$f = |\mathbf{F}|^2 = \mathbf{F}.\mathbf{F}, \tag{3.39}$$

and ask that the new vector $\mathbf{x}_{\text{new}}$ decreases f. The Newton step in Eq. (3.38) is actually a good candidate, as it ensures a descent for f. Indeed, we have

$$\nabla f.\delta\mathbf{x} = (\mathbf{F}.\mathbf{J}).(-\mathbf{J}^{-1}.\mathbf{F}) = -\mathbf{F}.\mathbf{F} < 0. \tag{3.40}$$

Thus the procedure will be the following: we always try the full Newton step Eq. (3.38), that will ensure quadratic convergence when we are close to the root. However, we check at each iteration that the new approximation really decreases the norm f of Eq. (3.39). If the proposed step does not represent a descent of f, we go back along the Newton direction until we have an acceptable step. Because the Newton step is a descent direction for f, we are bound to find an acceptable step. This method is essentially equivalent to minimising the norm f, by taking steps that bring $\mathbf{F}$ to zero. In practise the iteration will be

$$\mathbf{x}_{\mathrm{new}} = \mathbf{x}_{\mathrm{old}} + \lambda \mathbf{p}, \qquad 0 \leq \lambda \leq 1\,, \tag{3.41}$$

where $\mathbf{p}$ is the Newton step $\delta\mathbf{x}$ of Eq. (3.37) and λ is a parameter so that $f(\mathbf{x}_{\mathrm{old}} + \lambda\mathbf{p})$ has decreased sufficiently. We found in [13], a routine to achieve this task.

3.3 Chebyshev Expansion for M, Z And α

We have seen that when the function to be expanded has a general range $[a, b]$, then it is possible to make the change of variable of Eq. (3.24). In our case, the integral equations will be written in Euclidean space and the radial integration variable y, corresponding to momentum squared has the range $[0, \infty]$. Numerically, we will introduce an ultraviolet (UV) cut-off Λ^2 and a infrared (IR) cut-off ϵ^2. The UV cut-off is introduced to regularise the integrals, while the IR one serves a numerical purpose. The neglected part of the integration i.e. $\int_0^{\epsilon^2}$, has to be either evaluated analytically or shown to be negligible. In our study, we will choose an IR cut-off ϵ^2, such that the neglected contribution is indeed negligible. The phenomenon of dynamical symmetry breaking is in effect an IR phenomenon and it is thus essential to have enough grid points to show the evolution from a zero UV mass to a definite IR mass. We will thus not use a linear scale between ϵ^2 and Λ^2, but rather a logarithmic scale. The momentum p^2 will be changed to

$$t = \ln(p^2), \tag{3.42}$$

with range

$$[t_{\mathrm{min}}, t_{\mathrm{max}}] = [\ln(\epsilon^2), \ln(\Lambda^2)], \tag{3.43}$$

and we will map this range to $[-1, 1]$

$$s = \frac{t - \frac{1}{2}\left(t_{\max} + t_{\min}\right)}{\frac{1}{2}\left(t_{\max} - t_{\min}\right)}, \tag{3.44}$$

which can be rewritten

$$s = \frac{\ln(p^2/\Lambda\epsilon)}{\Lambda/\epsilon}. \tag{3.45}$$

The Chebyshev expansion for M, Z and α will be written

$$\begin{aligned} M(t = \ln(p^2)) &= \sum_{j=1}^{N_M}{}' a_j T_{j-1}(s), & (3.46) \\ Z(t) &= \sum_{j=1}^{N_Z}{}' b_j T_{j-1}(s), & (3.47) \\ \alpha(t) &= \sum_{j=1}^{N_\alpha}{}' c_j T_{j-1}(s), & (3.48) \end{aligned}$$

with s defined in Eq. (3.44). The advantage of using Chebyshev expansion is that we now have a smooth integrand and that we can compute the different functions at any point we want. We do not have to fix a set of grid-points, where all the functions are defined, thereby imposing on us a quadrature rule that has to use these grid points. Here, we can use any optimal set of grid-points which will be the zeros of the Legendre polynomials, i.e. we will use the powerful Gauss-Legendre quadrature formula to calculate the different integrals. This quadrature rule will be very efficient, because the integrand is now smooth.

3.4 Quadrature Rule

After having defined the Chebyshev expansion, we will need a quadrature rule to compute both the angular and radial integration. We will thus replace an integral by a summation

$$\int_a^b w(x) f(x) dx = \sum_{j=1}^{n} w_j f(x_j) + E_n[f], \tag{3.49}$$

where $w(x)$ is a weight function, w_j are the associated weights to compute the sum and x_j are a number a grid-points. $E_n[f]$ is the error generated when one replaces the integral by the sum. We would like to choose the weights w_j, such that the integral is exactly equal to the sum when the function is a polynomial of a certain order n. This happens when the n nodes x_j are the zeros of the polynomials $p_n(x)$, which satisfy an orthogonality relation with weight $w(x)$, i.e.

$$\int_a^b w(x)p_i(x)p_j(x)dx = \delta_{ij}. \tag{3.50}$$

The Gaussian quadrature with weight $w(x) = 1$ over the interval $[-1, 1]$ is called the Gauss-Legendre quadrature. In this scheme we have

$$\int_{-1}^1 f(x)dx = \sum_{j=1}^n w_j f(x_j) + E_n[f], \tag{3.51}$$

and the quadrature is exact if $f(x)$ is a polynomial of order n. The orthogonal polynomials with weight $w(x) = 1$ are the Legendre polynomials $P_n(x)$, which obey the following orthogonality relation

$$\int_{-1}^1 P_i(x)P_j(x)dx = \delta_{ij}\frac{2}{2j+1}. \tag{3.52}$$

They can be calculated explicitly by either the Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n, \tag{3.53}$$

or by recurrence

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{3.54}$$

The abscissas x_j will be the zeros of the Legendre polynomials $P_n(x)$ and their associated weight is

$$w_j = \frac{2}{\left(1-x_j^2\right)[P_n{'}(x_j)]^2}, \tag{3.55}$$

where the prime denotes a differentiation with respect to x. The error $E_n[f]$ is

$$E_n[f] = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi), \qquad -1 < \xi < 1. \tag{3.56}$$

In the case, where we are integrating over an interval $[a, b]$, we use the same integration points x_i and the same weights w_i, with the formula

$$\int_a^b f(x)dx = \frac{b-a}{2}\sum_{j=1}^n w_j\left(\frac{b+a}{2} + \frac{b-a}{2}x_j\right). \tag{3.57}$$

The error in this case will be

$$E_n[f] = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi), \qquad a < \xi < b. \tag{3.58}$$

3.5 Illustration

In this section we illustrate the method by a simple case where we have only one integral equation. In the bare vertex approximation, when the renormalisation function Z is set to unity, and the coupling function $\alpha(q^2)$ is set to its one loop expression, the mass equation satisfies the following integral equation

$$M(t) = \frac{3\alpha(\Lambda^2)}{2\pi^2} \int_{\ln(\epsilon^2)}^{\ln(\Lambda^2)} dt' \, \frac{y^2 M(t')}{y + M^2(t')} \int_0^{\pi} d\theta \, \frac{\sin^2\theta}{z\left(1 + \frac{N_f \alpha}{3\pi} \ln\left(\frac{\Lambda^2}{z}\right)\right)}, \tag{3.59}$$

with $x = e^t$, $y = e^{t'}$ and $z = x + y - 2\sqrt{xy}\cos\theta$.

We write $M(t)$ as a Chebyshev expansion

$$M(t) = \sum_{j=1}^{N_M}{}' a_j T_{j-1}(s), \tag{3.60}$$

with

$$s = \frac{t - \frac{1}{2}\left(t_{\max} + t_{\min}\right)}{\frac{1}{2}\left(t_{\max} - t_{\min}\right)}. \tag{3.61}$$

This expansion has N_M unknown expansion coefficients that we would like to determine. We will thus impose N_M constraints to find the a_j's. We impose that the integral equation has to be satisfied at N_M points $t_i, i = 1, N_M$

$$M(t_i) = \frac{3\alpha(\Lambda^2)}{2\pi^2} \int_{\ln(\epsilon^2)}^{\ln(\Lambda^2)} dt' \, \frac{y^2 M(t')}{y + M^2(t')} \int_0^{\pi} d\theta \, \frac{\sin^2\theta}{z_i\left(1 + \frac{N_f \alpha}{3\pi} \ln\left(\frac{\Lambda^2}{z_i}\right)\right)}, \qquad i = 1, \ldots, N_M, \tag{3.62}$$

where $M(t_i)$ is calculated using the Chebyshev expansion Eq. (3.60). These N_M abscissas $s_i \equiv s(t_i)$ will be chosen to be the zeros of the Chebyshev polynomial T_{N_M}, which are known to be

$$s_i = \cos\left(\frac{(i - 1/2)\,\pi}{N_M}\right), \qquad i = 1, \ldots, N_M. \tag{3.63}$$

The momentum on the t scale is thus

$$t_i = \ln(\Lambda\epsilon) + s_i \ln(\Lambda/\epsilon), \tag{3.64}$$

and on the linear scale we have

$$p_i^2 = x_i = \Lambda\epsilon \left(\frac{\Lambda}{\epsilon}\right)^{s_i}. \tag{3.65}$$

We now rewrite the N_M constraints Eq. (3.62), as sums using the Gauss-Legendre quadrature formula

$$M(t_i) = \frac{3\alpha(\Lambda^2)}{2\pi^2} \sum_{j=1}^{N_R} w_j^R \frac{y_j^2 M(t_j')}{y_j + M^2(t_j')} \sum_{k=1}^{N_\theta} w_k^\theta \frac{\sin^2\theta_k}{z_k\left(1 + \frac{N_f\alpha}{3\pi}\ln\left(\frac{\Lambda^2}{z_k}\right)\right)}, \qquad i = 1, \ldots, N_M, \tag{3.66}$$

with $y_j = e^{t_j'}$ and $z_k = x_i + y_j - 2\sqrt{x_i y_j}\cos\theta_k$. N_R is the number of nodes we use in the radial integration with weights w_j^R, while N_θ is the number of nodes for the angular integration with weight w_k^θ. To simplify the notation, we write

$$\Theta_{ik} = \sum_{k=1}^{N_\theta} w_k^\theta \frac{\sin^2\theta_k}{z_k\left(1 + \frac{N_f\alpha}{3\pi}\ln\left(\frac{\Lambda^2}{z_k}\right)\right)}. \tag{3.67}$$

We thus have

$$M(t_i) = \frac{3\alpha(\Lambda^2)}{2\pi^2} \sum_{j=1}^{N_R} w_j^R \frac{y_j^2 M(t_j')\Theta_{ik}}{y_j + M^2(t_j')} \qquad i = 1, \ldots, N_M. \tag{3.68}$$

These N_M algebraic equations form an (almost) algebraic non-linear system that we solve by the global convergent method described previously. Because the equation is non-linear, the expansion coefficients a_j cannot be taken outside the radial integral, and we have to recompute the radial integral after each iteration. The system is thus not an algebraic system, it would have been had we been able to rewrite the non-linear system as a sum of constant coefficients times unknown variables. We define the vector $\mathbf{F}$ to be zeroed

$$\mathbf{F} = \begin{pmatrix} F_1 \\ \vdots \\ F_{N_M}, \end{pmatrix}, \tag{3.69}$$

whose components F_i are

$$F_i = \sum_{j=1}^{N_M}{}' a_j T_{j-1}(s_i) - \frac{3\alpha(\Lambda^2)}{2\pi^2} \sum_{j=1}^{N_R} w_j^R \frac{y_j^2 M(t_j')\Theta_{ik}}{y_j + M^2(t_j')}, \qquad i = 1, \ldots, N_M. \tag{3.70}$$

We now have to define the Jacobian $\mathbf{J}$. It is given by

$$J_{ij}(\mathbf{a}) = \frac{\partial F_i(\mathbf{a})}{\partial a_j}, \tag{3.71}$$

whose explicit representation is

$$J_{ij}(\mathbf{a}) = T_j(s_i) - \frac{1}{2}\delta_{1j} - \frac{3\alpha(\Lambda^2)}{2\pi^2} \sum_{j=1}^{N_R} w_j^R \left[T_j(r_j) - \frac{1}{2}\delta_{1k}\right] \frac{y_j^2 \Theta_{ik}\left(y_j - M^2(t_j')\right)}{\left(y_j + M^2(t_j')\right)^2}, \tag{3.72}$$

where r_j maps t_j' to the interval $[-1,1]$. To write this expression we have also used

$$\frac{\partial M(t_i)}{\partial a_j} = T_j(s_i) - \frac{1}{2}\delta_{1j}. \tag{3.73}$$

The Newton iteration is

$$\mathbf{a}_{n+1} = \mathbf{a}_n - \delta\mathbf{a}, \tag{3.74}$$

with $\delta\mathbf{a}$ satisfying

$$\mathbf{J}(\mathbf{a}_n)\delta\mathbf{a} = \mathbf{F}(\mathbf{a}_n), \tag{3.75}$$

that is solved by using the explicit expression Eq. (3.70-3.72).

Once we have determined the expansion coefficients, we can compute the mass function $M(x)$ for any values we want using the Clenshaw's recurrence formula Eq. (3.19). We postpone to the next chapter the actual implementation of this method for QED in the MR scheme, first when the coupling function is set to its one loop expression and then we apply it to the full system (M, Z, α).

Chapter 4

Numerical Solution of the QED System in the MR Scheme

In this chapter, we apply the numerical method developed in the previous chapter to the system of equations Eq. (2.154-2.156), which are the SD equations for QED in the MR scheme. Before solving the full system, we illustrate what happens when we make further approximations. We first present the so-called case of quenched QED, which is obtained by approximating the coupling function $\alpha(q^2)$ by its constant bare value at the cut-off Λ^2. We then show what happens when we set the coupling function to its one loop expression and finally solve the full system Eq. (2.154-2.156).

4.1 Quenched QED in the MR Scheme

The equations for quenched QED are obtained, when we set the coupling function to a constant, i.e.

$$\alpha(q^2) = \alpha(\Lambda^2)\,, \qquad \text{quenched case.} \tag{4.1}$$

This amounts to neglecting fermion loops in the photon propagator.

In order to treat the quenched case, we start from the Minkowski equations, keeping the gauge parameter ξ_0

$$\frac{1}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2) + \frac{i\alpha(\Lambda^2)}{4\pi^3 p^2}\int d^4k\,\frac{1}{Z_R(k^2,\mu^2)}\frac{1}{k^2-M^2(k^2)}\frac{K_G+K_{\xi_0}}{(p-k)^2} \tag{4.2}$$

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2)m_0(\Lambda^2) - \frac{i\alpha(\Lambda^2)}{4\pi^3}\int d^4k \frac{M(k^2)}{Z_R(k^2,\mu^2)}\frac{1}{k^2 - M^2(k^2)}\frac{3+\xi_0}{(p-k)^2}, \tag{4.3}$$

with

$$K_G = \frac{2}{(p-k)^2}\left(k^2p^2 - (k.p)^2\right) - 3p.k, \tag{4.4}$$

$$K_{\xi_0} = \frac{\xi_0}{(p-k)^2}\left(k^2+p^2)(p.k) - 2p^2k^2\right). \tag{4.5}$$

We first consider the equation satisfied by $M(p^2)/Z_R(p^2,\mu^2)$. We do not at this point perform a Wick rotation, but note that the kernel $1/(p-k)^2$ has a Fourier transform that can be computed analytically, which corresponds to a massless scalar particle

$$\int \frac{d^4k}{2\pi^4}\frac{1}{k^2}e^{ik.w} = \frac{i}{4\pi^2}\frac{1}{w^2 - i\epsilon}. \tag{4.6}$$

Also the integrand in Eq. (4.3) is the product of a function of k^2 multiplied by a function of $(p-k)^2$ only i.e. the integrand is a direct product, and the Fourier transform of a direct product is the product of the Fourier transforms. We therefore take a Fourier transform from momentum space p^2 to co-ordinate space w. If we write

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)} \equiv B(p^2,\mu^2), \tag{4.7}$$

$$\frac{1}{Z_R(p^2,\mu^2)} \equiv A(p^2,\mu^2), \tag{4.8}$$

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)}\frac{1}{k^2 - M^2(k^2)} \equiv \sigma_B(k^2,\mu^2), \tag{4.9}$$

then in co-ordinate space we have

$$B(w,\mu^2) = m_0 Z_2(\mu^2,\Lambda^2)\delta^4(w) - (3+\xi_0)i4\pi\alpha(\Lambda^2)\sigma_B(w,\mu^2)\frac{i}{4\pi^2}\frac{1}{w^2 - i\epsilon}, \tag{4.10}$$

or

$$w^2\left(B(w,\mu^2) - m_0 Z_2(\mu^2,\Lambda^2)\delta^4(w)\right) = \frac{(3+\xi_0)}{4\pi^2}4\pi\alpha(\Lambda^2)\sigma_B(w,\mu^2). \tag{4.11}$$

Because the factor w^2 is multiplying the delta function, $m_0 Z_2(\mu^2, \Lambda^2) w^2 \delta^4(w)$ disappears and we lose any dependence on the mass scale μ^2. This is understandable and natural since quenched QED has no intrinsic mass scale. By Fourier transforming back to momentum space, we obtain the following differential equation

$$xB''(x) + 2B'(x) + \frac{(3+\xi_0)\alpha(\Lambda^2)}{4\pi} \frac{B(x)}{x - (B(x)/A(x))^2} = 0, \tag{4.12}$$

with $x = p^2$ and the prime denotes differentiation with respect to x. This equation is to be contrasted with the one we can derive in the bare vertex approximation

$$xB''(x) + 2B'(x) + \frac{(3+\xi_0)\alpha(\Lambda^2)}{4\pi} \frac{1}{A(x)} \frac{B(x)/A(x)}{x - (B(x)/A(x))^2} = 0, \tag{4.13}$$

which differs from Eq. (4.12) by its $A(x)$ dependence. In the often encountered $A \equiv 1$ approximation, Eq. (4.12) and Eq. (4.13) are the same. It should be noted here that Eq. (4.12) is valid in the whole $x = p^2$ Minkowski plane. Had we performed a Wick rotation, we would have in this particular case derived the same equation in the Euclidean p^2 space, but the Euclidean equation would need analytical continuation to negative p^2, obtained from the consideration of the kernel of integral equation Eq. (4.3). Kugo et al. [14], have shown that the analytically continued equation take the same form. Our result in Minkowski space spares us the need for analytical continuation. However, this works here because the kernel $1/(p-k)^2$ is quite simple and its singularity structure is known. Instead of taking Fourier transform, we could have differentiated Eq. (4.3) twice with respect to the momentum variable p^μ and would have obtained the same equation using the relation

$$\Box_q \frac{1}{q^2} = -i4\pi^2 \frac{1}{w^2}, \tag{4.14}$$

where q is the momentum variable and w the spacetime variable.

Unfortunately, for the equation Eq. (4.2) satisfied by $Z_R(x, \mu^2)$, the form of the kernel does not allow us to treat this equation in the same way as we did for Eq. (4.3). We will thus follow the usual procedure of Wick rotation. Because of the absence of the coupling function in the angular integral, the angular integration is computable analytically. After introducing the Heaviside step function (or theta function), $\Theta(x)$ we have

$$
\begin{aligned}
\frac{1}{Z_R(x,\mu^2)} &= Z_2(\mu^2,\Lambda^2) \\
&-\frac{\alpha(\Lambda^2)}{2\pi^2 x}\int dy\,\frac{y/Z_R(y,\mu^2)}{y+M^2(y)}\int d\theta\,\sin^2\theta\left(\frac{2xy\sin^2\theta}{z^2}-\frac{3\sqrt{xy}\cos\theta}{z}\right) \\
&+\frac{\alpha(\Lambda^2)\xi_0}{4\pi}\int dy\,\frac{1/Z_R(y,\mu^2)}{y+M^2(y)}\left(\frac{y^2}{x^2}\Theta(x-y)+\Theta(y-x)\right). \qquad (4.15)
\end{aligned}
$$

We use now [3]

$$
\int_0^\pi d\theta\,\frac{\sin^4\theta}{z^2} = \frac{3\pi}{8}\left[\frac{\Theta(x-y)}{x^2}+\frac{\Theta(y-x)}{y^2}\right], \qquad (4.16)
$$

$$
\int_0^\pi d\theta\,\frac{\sin^2\theta\cos\theta}{z} = \frac{\pi}{4\sqrt{xy}}\left[\frac{y}{x}\Theta(x-y)+\frac{x}{y}\Theta(y-x)\right], \qquad (4.17)
$$

and obtain

$$
\frac{1}{Z_R(x,\mu^2)} = Z_2(\mu^2,\Lambda^2)+\frac{\alpha(\Lambda^2)\xi_0}{4\pi}\int_0^\infty dy\,\frac{1/Z_R(y,\mu^2)}{y+M^2(y)}\left(\frac{y^2}{x^2}\Theta(x-y)+\Theta(y-x)\right), \qquad (4.18)
$$

where we notice that the only contribution comes from the gauge fixing part of the photon propagator. The Θ functions in the integrand restrict the range of the variable y and Eq. (4.18) can be rewritten

$$
\begin{aligned}
\frac{1}{Z_R(x,\mu^2)} &= Z_2(\mu^2,\Lambda^2)+\frac{\alpha(\Lambda^2)\xi_0}{4\pi}\int_0^x dy\,\frac{1/Z_R(y,\mu^2)}{y+M^2(y)}\frac{y^2}{x^2} \qquad (4.19)\\
&+\frac{\alpha(\Lambda^2)\xi_0}{4\pi}\int_x^\infty dy\,\frac{1/Z_R(y,\mu^2)}{y+M^2(y)}.
\end{aligned}
$$

We use the notation of Eq. (4.8) i.e. $A \equiv 1/Z_R$, and differentiate the last equation with respect to x. We obtain

$$
x^3 A'(x) = -\frac{2\alpha(\Lambda^2)\xi_0}{4\pi}\int_0^x dy\,\frac{1/Z_R(y,\mu^2)}{y+M^2(y)}y^2. \qquad (4.20)
$$

If we differentiate once more with respect to x, we obtain

$$
\left[x^3 A'(x)\right]' = -\frac{\alpha(\Lambda^2)\xi_0}{2\pi}x^2\frac{A(x)}{x+M^2(x)}, \qquad (4.21)
$$

which can be rewritten as

$$xA''(x) + 3A'(x) + \frac{\alpha(\Lambda^2)\xi_0}{2\pi}\frac{A(x)}{x + B^2(x)/A^2(x)} = 0. \tag{4.22}$$

This equation with Eq. (4.12) have to be solved together to give the solution for quenched QED in the MR scheme. We will here focus only on the domain where x is much bigger than $B^2(x)/A^2(x)$. In this case, we can rewrite Eq. (4.12,4.22) as

$$x^2A''(x) + 3xA'(x) + \frac{\alpha(\Lambda^2)\xi_0}{2\pi}A(x) = 0 \tag{4.23}$$

$$x^2B''(x) + 2xB'(x) + \frac{(3+\xi_0)}{4\pi}\alpha(\Lambda^2)B(x) = 0. \tag{4.24}$$

We use the following ansatz for the solution of the above system

$$B(x) = x^{-s} \tag{4.25}$$

$$A(x) = x^{\nu}, \tag{4.26}$$

and obtain the following relations

$$\nu^2 + 2\nu + \frac{\alpha(\Lambda^2)\xi_0}{2\pi} = 0 \tag{4.27}$$

$$s^2 - s + \frac{(3+\xi_0)}{4\pi}\alpha(\Lambda^2) = 0. \tag{4.28}$$

The solutions for s and ν are

$$\nu = -1 \pm \sqrt{1 - \alpha\xi_0/2\pi} \tag{4.29}$$

$$s = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \alpha/\alpha_c}, \tag{4.30}$$

where $\alpha(\Lambda^2)$ has been written α and

$$\alpha_c = \frac{\pi}{3+\xi_0} \tag{4.31}$$

is the critical coupling. For $\alpha < \alpha_c$, we are in the sub-critical regime and the exponent s is real. For $\alpha > \alpha_c$, s is complex and the function $B(x)$ or the mass function $M(x)$ has an oscillatory behaviour, which indicates a new phase in quenched QED. We shall not solve the system numerically here but rather look at the case of QED, where the coupling is approximated by its one loop value.

4.2 A Calculation in Minkowski Space

We compare the Minkowski space calculation to the Euclidean formulation. We work in the bare vertex approximation and in the non-local gauge $\xi_0 = G(q^2)$, where $G(q^2)$ is the photon dressing function. Even though our application will be for the case of quenched QED.

The propagators are

$$\begin{aligned} S(p) &= \frac{Z(p^2)}{p^2 - M^2(p^2)}\left(\not p + M(p^2)\right), \\ D_{\mu\nu}(q) &= -\frac{G(q^2)}{q^2}\left(g_{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \xi_0 \frac{q^\mu q^\nu}{q^4}. \end{aligned}$$

In Minkowski space, the SD equations in the bare vertex approximation are

$$\frac{M(p^2)}{Z(p^2)} = m_0 - \frac{ie_0^2}{(2\pi)^4}\int d^4k\, \sigma_M(k^2)\frac{4G(q^2)}{q^2}, \tag{4.32}$$

$$\begin{aligned} \frac{1}{Z(p^2)} &= 1 + \frac{ie_0^2}{p^2(2\pi)^4}\int d^4k\, \sigma_Z(k^2) \\ &\quad \times G(q^2)\left\{\left[\frac{2p^2k^2 - 2(p.k)^2}{q^4} - 3\frac{p.k}{q^2}\right] + \left[\frac{(p^2+k^2)p.k - 2p^2k^2}{q^4}\right]\right\}, \end{aligned} \tag{4.33}$$

$$\sigma_M(k^2) = \frac{Z(k^2)M(k^2)}{k^2 - M^2(k^2)},$$

$$\sigma_Z(k^2) = \frac{Z(k^2)}{k^2 - M^2(k^2)}.$$

The term in square bracket representing the contribution of the gauge parameter ξ_0 can be rewritten as follows

$$\begin{aligned} (p^2 + k^2)(p.k) &= \left[q^2 + 2(p.k)\right](p.k) \\ &= q^2(p.k) + 2(p.k)^2. \end{aligned}$$

The equation for $Z(p^2)$ becomes

$$\frac{1}{Z(p^2)} = 1 - \frac{2ie_0^2}{p^2(2\pi)^4}\int d^4k\, \sigma_Z(k^2)G(q^2)\frac{p.k}{q^2}. \tag{4.34}$$

We now consider the case of quenched QED and thus fix $G(q^2) = 1$. The system of integral equations in Minkowski space is now

$$\frac{M(p^2)}{Z(p^2)} = m_0 - \frac{4ie_0^2}{(2\pi)^4} \int d^4k\, \sigma_M(k^2)\frac{1}{q^2}\,, \tag{4.35}$$

$$\frac{1}{Z(p^2)} = 1 - \frac{2ie_0^2}{p^2(2\pi)^4} \int d^4k\, \sigma_Z(k^2)\frac{p.k}{q^2}\,. \tag{4.36}$$

As we have seen in the previous section the integral equation for $B(p^2) = M(p^2)/Z(p^2)$ can be converted into a differential equation which is

$$xB''(x) + 2B'(x) + \frac{\alpha(\Lambda^2)}{\pi}\sigma_Z(x) = 0, \tag{4.37}$$

where $x = p^2$ is the Minkowski squared momentum.

The equation for $A(p^2) = (1/Z - 1)$ is more involved since the kernel involves the momenta p but not only through the combination $(p - k)$. We rewrite Eq. (4.36) in the following way

$$\begin{aligned} p^2A(p^2) &= -\lambda\int d^4k\, \sigma_Z(k^2)\frac{p.k}{q^2}, \\ p_ap_aA(p^2) &= -\lambda p_a\int d^4k\, k_a\sigma_Z(k^2)\frac{1}{q^2}. \end{aligned} \tag{4.38}$$

We can therefore eliminate the term p_a on the right hand side. we have

$$A_a(p^2) = -\lambda\int d^4k\, \sigma_Z^a(k^2)\frac{1}{q^2}, \tag{4.39}$$

with $A_a(p^2) = p_aA(p^2)$ and $\sigma_Z^a(p^2) = k_a\sigma_Z(p^2)$. If we now take the Fourier transform to space time variable w then Fourier transform back to momentum space, we obtain the following differential equation

$$\Box_k\left[k_aA(k^2)\right] = \frac{i\lambda}{4\pi^2}k_a\sigma_Z(k^2), \tag{4.40}$$

or

$$xA''(x) + 3A'(x) + \frac{\alpha(\Lambda^2)}{2\pi}\sigma_Z(x) = 0, \qquad x = k^2\,, \tag{4.41}$$

which, for $x \leq 0$ is the same as we would have derived if we had performed a Wick rotation.

In this rather simple example, we have seen that it is possible to write a differential equation for the mass function $M(x)$ and the dressing function $Z(x)$ that is valid in the whole $x = p^2$ line, without resorting to the Wick rotation. The equations derived after Wick rotation, i.e. those for $x \geq 0$ have the same form. It therefore shows us that the analytical continuation to negative $x = p^2$ is in this case straightforward. However it is known, that in the complex p^2 plane, the mass function $M(p^2)$ derived by performing a Wick rotation has branch points [25], whose location depend on the value of the bare coupling $\alpha(\Lambda^2)$. These branch points should thus play a role when one performs a Wick rotation. It is thus rather strange, that we find the same equation for $x \leq 0$. The role played by these singularities is still not very clear and it seems that it is interpreted in a way that is convenient by the context in which they are found (as we have already mentioned in chapter 2, where we first introduced the Wick rotation).

4.3 *MR* Scheme at One Loop Approximation

In this section, we solve the *MR* scheme *QED* equations in the one loop approximation. This simplification allows us to treat the system as a nonlinear system for the expansion coefficients of the functions $M(x)$ and $Z_R(x)$. The coupling function $\alpha(x)$ is

$$\alpha_{1\ell}(x) = \frac{\alpha(\Lambda^2)}{1 + \frac{N_f \alpha(\Lambda^2)}{3\pi} \ln\left(\frac{\Lambda^2}{x}\right)}, \tag{4.42}$$

where Λ^2 is the cut-off that we will choose to be

$$\Lambda^2 = 10^{10}, \tag{4.43}$$

so as to make comparison with earlier work [3]. We will thus plot for example the mass function $M(p^2)$ with

$$10^{-4} \leq p^2 \leq 10^{10}, \tag{4.44}$$

and show the p^2 axis with these units, which in the case of *QCD* would typically be GeV units. Once we have fixed a value to the cut-off Λ^2, the mass function for another cut-off $\Lambda_\lambda^2 = \lambda\Lambda^2$ is obtained easily since the equation for the mass is invariant under the following transformation

$$
\begin{aligned}
x &\rightarrow \lambda x, && (4.45)\\
M &\rightarrow \sqrt{\lambda} M. && (4.46)\\
& && (4.47)
\end{aligned}
$$

The system to be solved is

$$
\frac{M(x)}{Z_R(x,\mu^2)} = m_0 Z_2(\mu^2,\Lambda^2) + \frac{3}{2\pi^2}\int dy\, \frac{M(y)}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\int d\theta\, \sin^2\theta \frac{\alpha_{1\ell}(z)}{z}, \tag{4.48}
$$

$$
\begin{aligned}
\frac{1}{Z_R(x,\mu^2)} &= Z_2(\mu^2,\Lambda^2) + \frac{1}{2\pi^2}\int dy\, \frac{1}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\\
&\times \int d\theta\, \sin^2\theta \frac{\alpha_{1\ell}(z)}{x}\left[\frac{3\sqrt{xy}\cos\theta}{z} - \frac{2xy\sin^2\theta}{z^2}\right], && (4.49)
\end{aligned}
$$

with $x = p^2$, $y = k^2$, $z = (p-k)^2 = x + y - 2\sqrt{xy}\cos\theta$ and $\alpha_{1\ell}(z)$ is the one-loop expression Eq. (4.42). These equations still contain the unknown renormalisation function $Z_2(\mu^2, \Lambda^2)$, which can be removed if we rewrite the system for an arbitrary momentum x that we will choose to be $x = \mu^2$. Because of the renormalisation condition

$$
Z_R(\mu^2,\mu^2) = 1, \tag{4.50}
$$

we have

$$
M(\mu^2) = m_0 Z_2(\mu^2,\Lambda^2) + \frac{3}{2\pi^2}\int dy\, \frac{M(y)}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\int d\theta\, \sin^2\theta \frac{\alpha_{1\ell}(z_\mu)}{z_\mu}, \tag{4.51}
$$

$$
\begin{aligned}
1 &= Z_2(\mu^2,\Lambda^2) + \frac{1}{2\pi^2}\int dy\, \frac{1}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\\
&\times \int d\theta\, \sin^2\theta \frac{\alpha_{1\ell}(z_\mu)}{\mu^2}\left[\frac{3\sqrt{xy}\cos\theta}{z_\mu} - \frac{2xy\sin^2\theta}{z_\mu^2}\right], && (4.52)
\end{aligned}
$$

where $z_\mu = \mu^2 + y - 2\mu\sqrt{y}\cos\theta$. If we subtract the two system of equations and write $M(\mu^2) = M_\mu$ we obtain

$$\frac{M(x)}{Z_R(x,\mu^2)} = M_\mu + \frac{3}{2\pi^2}\int dy\,\frac{M(y)}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\int d\theta\,\sin^2\theta\left[\frac{\alpha_{1\ell}(z)}{z} - x \leftrightarrow \mu^2\right], \tag{4.53}$$

$$\frac{1}{Z_R(x,\mu^2)} = 1 + \frac{1}{2\pi^2}\int dy\,\frac{1}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}$$
$$\int d\theta\,\sin^2\theta\left\{\alpha_{1\ell}(z)\left[\frac{3\sqrt{y/x}\cos\theta}{z} - \frac{2y\sin^2\theta}{z^2}\right] - x \leftrightarrow \mu^2\right\}. \tag{4.54}$$

The system, we have just written is for the massive case $m_0 \neq 0$. In the massless case, the equation for M/Z does not need subtraction and we can thus replace Eq. (4.53) by

$$\frac{M(x)}{Z_R(x,\mu^2)} = \frac{3}{2\pi^2}\int dy\,\frac{M(y)}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\int d\theta\,\sin^2\theta\frac{\alpha_{1\ell}(z)}{z}. \tag{4.55}$$

In order to solve this system, we first make an expansion for the two functions $M(t)$ and $Z_R(t,\mu^2)$, where the argument $t = \ln(x)$ has the range

$$[t_{\rm min}, t_{\rm max}] = [\ln(\epsilon^2), \ln(\Lambda^2)]. \tag{4.56}$$

We have

$$M(t) = \sum_{j=1}^{N_M}{}' a_j T_{j-1}(s), \tag{4.57}$$

$$Z(t) = \sum_{j=1}^{N_Z}{}' b_j T_{j-1}(s), \tag{4.58}$$

with

$$s = \frac{t - \frac{1}{2}(t_{\rm max} + t_{\rm min})}{t_{\rm max} - t_{\rm min}}, \tag{4.59}$$

where we have chosen N_M external momenta where we impose that Eq. (4.55) has to be satisfied and N_Z momenta where we impose that Eq. (4.54) has to be satisfied. For the radial integration, we split the integrals at $y = x$ so as to avoid the integration over the kink. We have here to notice that the angular integrals depends on the variable $z = (p-k)^2$. For a squared momentum range $[\epsilon^2, \Lambda^2]$, the variable z has range $[0, 4\Lambda^2]$ and we thus have to extrapolate the coupling function outside its range of definition. We will choose to extrapolate the coupling in a way that maintains its continuity, which is vital for good accuracy in the numerical calculation. We choose

$$\alpha_{1\ell}(x) = \alpha_{1\ell}(\epsilon^2), \qquad x < \epsilon^2, \tag{4.60}$$

$$\alpha_{1\ell}(x) = \alpha_{1\ell}(\Lambda^2), \qquad x > \Lambda^2 . \tag{4.61}$$

Using the Gauss-Legendre quadrature rule, we rewrite the integrals as sums. The angular integrals depend only on the coupling function $\alpha(x)$ and can thus be computed once and for all and stored. In the massless case, the two angular integrals are

$$\begin{aligned}\Theta_{M(t_i,t_j)} &= \int d\theta \, \sin^2\theta_k \frac{\alpha_{1\ell}(z)}{z} \\ &= \sum_{k=1}^{N\theta} w_k \sin^2\theta_k \frac{\alpha_{1\ell}(z_k)}{z_k}\end{aligned} \tag{4.62}$$

$$\begin{aligned}\Theta_{Z(t_i,t_j)} &= \int d\theta \, \sin^2\theta \left\{ \alpha_{1\ell}(z) \left[\frac{3\sqrt{y/x}\cos\theta}{z} - \frac{2y\sin^2\theta}{z^2} \right] - x \leftrightarrow \mu^2 \right\} \\ &= \sum_{k=1}^{N\theta} \sin^2\theta \left\{ \alpha_{1\ell}(z_k) \left[\frac{3\sqrt{y_j/x_i}\cos\theta_k}{z_k} - \frac{2y_j\sin^2\theta_k}{z_k^2} \right] - t_i \leftrightarrow \ln(\mu^2) \right\},\end{aligned} \tag{4.63}$$

with $t_i = \ln(x_i)$, $t_j = \ln(y_j)$ and $z_k = x_i + y_j - 2\sqrt{x_i y_j}\cos\theta_k$. N_θ is the number of quadrature points we choose for the angular integration. We now have a system of $N_M + N_Z$ non-linear equations to determine the N_M a_j expansion coefficients of $M(t)$ and the N_Z b_j expansion coefficients of $Z_R(t, \mu^2)$. The vector function $\mathcal{F}$ to be zeroed and defined as

$$\mathcal{F} = (F_{1,1}, \ldots, F_{1,N_M}, F_{2,1}, \ldots, F_{2,N_Z}), \tag{4.64}$$

has the explicit components

$$F_{1,i} = \frac{M(t_i)}{Z_R(t_i)} - \frac{3}{2\pi^2} \sum_{j=1}^{N_R} w_j \frac{M(t_j)}{Z_R(t_j)} \frac{y_j^2}{y_j^2 + M^2(t_j)} \Theta_M(t_i, t_j), \qquad i = 1, \ldots, N_M \tag{4.65}$$

$$\begin{aligned} F_{2,i} &= \frac{1}{Z_R(t_i)} - 1 \\ &- \frac{1}{2\pi^2}\sum_{j=1}^{N_R} w_j \frac{1}{Z_R(t_j)}\frac{y_j^2}{y_j^2+M^2(t_j)}\left(\frac{1}{x_i}\Theta_Z(t_i,t_j) - x \leftrightarrow \mu^2\right), \quad i=1,\ldots,N_Z, \end{aligned} \tag{4.66}$$

where $N_R = N_{R1} + N_{R2}$ is the total number of integration points used in the Gauss formula to compute the split radial integrals. We now have to define the Jacobian J in order to implement the Newton iteration. It is

$$J_{ij} = \left(\begin{array}{c|c} \frac{\partial F_{1,i}}{\partial a_j} & \frac{\partial F_{1,i}}{\partial b_j} \\ \hline \frac{\partial F_{2,i}}{\partial a_j} & \frac{\partial F_{2,i}}{\partial b_j} \end{array}\right), \tag{4.67}$$

with explicit expressions

$$\begin{aligned} \frac{\partial F_{1,i}}{\partial a_j}(\mathbf{a},\mathbf{b}) &= \frac{\partial}{\partial a_j}\left(\frac{M(t_i)}{Z_R(t_i)} - \frac{3}{2\pi^2}\sum_{j=1}^{N_R} w_j \frac{M(t_j)}{Z_R(t_j)}\frac{y_j^2}{y_j^2+M^2(t_j)}\Theta_M(t_i,t_j)\right) \\ &= \frac{\left[T_j(s_i) - \frac{1}{2}\delta_{1j}\right]}{Z_R(t_i)} \\ &- \frac{3}{2\pi^2}\sum_{j=1}^{N_R} w_j \frac{y_j^2 Z_R(t_j)\left(y_j^2 - M^2(t_j)\right)\left[T_j(r_i) - \frac{1}{2}\delta_{1j}\right]}{\left(y_j^2+M^2(t_j)\right)^2}\Theta_M(t_i,t_j), \end{aligned} \tag{4.68}$$

$$\begin{aligned} \frac{\partial F_{1,i}}{\partial b_j}(\mathbf{a},\mathbf{b}) &= \frac{\partial}{\partial b_j}\left(\frac{M(t_i)}{Z_R(t_i)} - \frac{3}{2\pi^2}\sum_{j=1}^{N_R} w_j \frac{M(t_j)}{Z_R(t_j)}\frac{y_j^2}{y_j^2+M^2(t_j)}\Theta_M(t_i,t_j)\right) \\ &= -\frac{M(t_i)\left[T_j(s_i) - \frac{1}{2}\delta_{1j}\right]}{Z_R^2(t_i)} \\ &- \frac{3}{2\pi^2}\sum_{j=1}^{N_R} w_j \frac{y_j^2\left[T_j(r_i) - \frac{1}{2}\delta_{1j}\right]M(t_j)}{\left(y_j^2+M^2(t_j)\right)^2}\Theta_M(t_i,t_j), \end{aligned} \tag{4.69}$$

$$\frac{\partial F_{2,i}}{\partial a_j}(\mathbf{a},\mathbf{b}) = \frac{\partial}{\partial a_j}\left(\frac{1}{Z_R(t_i)} - Z_2(\mu^2,\Lambda^2) - \frac{1}{2\pi^2 x_i}\sum_{j=1}^{N_R} w_j \frac{1}{Z_R(t_j)}\frac{y_j^2\Theta_Z(t_i,t_j)}{y_j^2+M^2(t_j)}\right)$$

$$= \frac{1}{\pi^2 x_i} \sum_{j=1}^{N_R} w_j y_j^2 \frac{1}{Z(t_j)} \frac{\left[T_j(r_i) - \frac{1}{2}\delta_{1j}\right] M(t_j)}{\left(y_j^2 + M^2(t_j)\right)^2} \Theta_Z(t_i, t_j)\,, \tag{4.70}$$

$$\begin{aligned} \frac{\partial F_{2,i}}{\partial b_j}(\mathbf{a},\mathbf{b}) &= \frac{\partial}{\partial b_j}\left(\frac{1}{Z_R(t_i)} - Z_2(\mu^2,\Lambda^2) - \frac{1}{2\pi^2 x_i}\sum_{j=1}^{N_R} w_j \frac{1}{Z_R(t_j)} \frac{y_j^2 \Theta_Z(t_i,t_j)}{y_j^2 + M^2(t_j)}\right) \\ &= -\frac{\left[T_j(s_i) - \frac{1}{2}\delta_{1j}\right]}{Z_R^2(t_i)} \\ &\quad + \frac{1}{2\pi^2 x_i}\sum_{j=1}^{N_R} w_j \frac{1}{Z_R^2(t_j)} \frac{\left[T_j(r_i) - \frac{1}{2}\delta_{1j}\right] y_j^2}{y_j^2 + M^2(t_j)} \Theta_Z(t_i,t_j). \end{aligned} \tag{4.71}$$

The analytical computation of this Jacobian is done using the non-subtracted expression still involving the renormalisation function $Z_2(\mu^2, \Lambda^2)$, which does not depend on x nor does it involve the integration over y.

The Newton vector increment $\delta\mathbf{x}$ satisfies the following matrix equation

$$J\left(\mathbf{a}_n, \mathbf{b}_n\right)\delta\mathbf{x} = \mathbf{F}\left(\mathbf{a}_n, \mathbf{b}_n\right), \tag{4.72}$$

where n is the number of the Newton iteration. Explicitly we have

$$\sum_{j=1}^{N_M} \frac{\partial F_{1,i}(\mathbf{a}_n, \mathbf{b}_n)}{\partial a_j}(\delta_{a,n+1})_j + \sum_{j=1}^{N_Z} \frac{\partial F_{1,i}(\mathbf{a}_n, \mathbf{b}_n)}{\partial b_j}(\delta_{b,n+1})_j = F_{1,i}(\mathbf{a}_n, \mathbf{b}_n),\ i \le N_M, \tag{4.73}$$

$$\sum_{j=1}^{N_M} \frac{\partial F_{2,i}(\mathbf{a}_n, \mathbf{b}_n)}{\partial a_j}(\delta_{a,n+1})_j + \sum_{j=1}^{N_Z} \frac{\partial F_{2,i}(\mathbf{a}_n, \mathbf{b}_n)}{\partial b_j}(\delta_{b,n+1})_j = F_{2,i}(\mathbf{a}_n, \mathbf{b}_n),\ i \le N_Z. \tag{4.74}$$

The solution of this linear system gives the Newton increment $\delta\mathbf{x} = (\delta_{\mathbf{a},n+1}, \delta_{\mathbf{b},n+1})$ which yields the new approximation by the following Newton relations

$$\begin{aligned} \mathbf{a}_{n+1} &= \mathbf{a}_n - \delta_{\mathbf{a},n+1}, \\ \mathbf{b}_{n+1} &= \mathbf{b}_n - \delta_{\mathbf{b},n+1}\,. \end{aligned} \tag{4.75}$$

To start with, we have to choose a guess for the function M and Z_R, which provides us with $\mathbf{a}_0$ and $\mathbf{b}_0$. In the program, we choose $N_M = N_Z = 50$. The Gauss-Legendre quadrature are performed with $N_\theta = 32$ angular points and $N_R = 120 + 120$ radial points. In order to make comparisons with previous work, we choose the subtraction point μ^2 to be our cut-off Λ^2, i.e.

$$\mu^2 = \Lambda^2. \tag{4.76}$$

For $N_F = 1$, we find the critical coupling $\alpha_{1\ell C} = \alpha_{1\ell C}(\Lambda^2)$ to be

$$\alpha_{1\ell C} = 2.084312\,. \tag{4.77}$$

We show in Fig. (4.1) and Fig. (4.2) the typical behaviour for the mass function $M(p^2)$ and the dressing function $Z_R(p^2, \mu^2)$ renormalised at $\mu^2 = \Lambda^2$ for $N_f = 1$ and for three different values of the coupling $\alpha(\Lambda^2)$. We also plot in Fig. (4.3) the infrared mass $M(0, N_f = 1)$ as a function of the bare coupling $\alpha_{1\ell}(\Lambda^2)$.

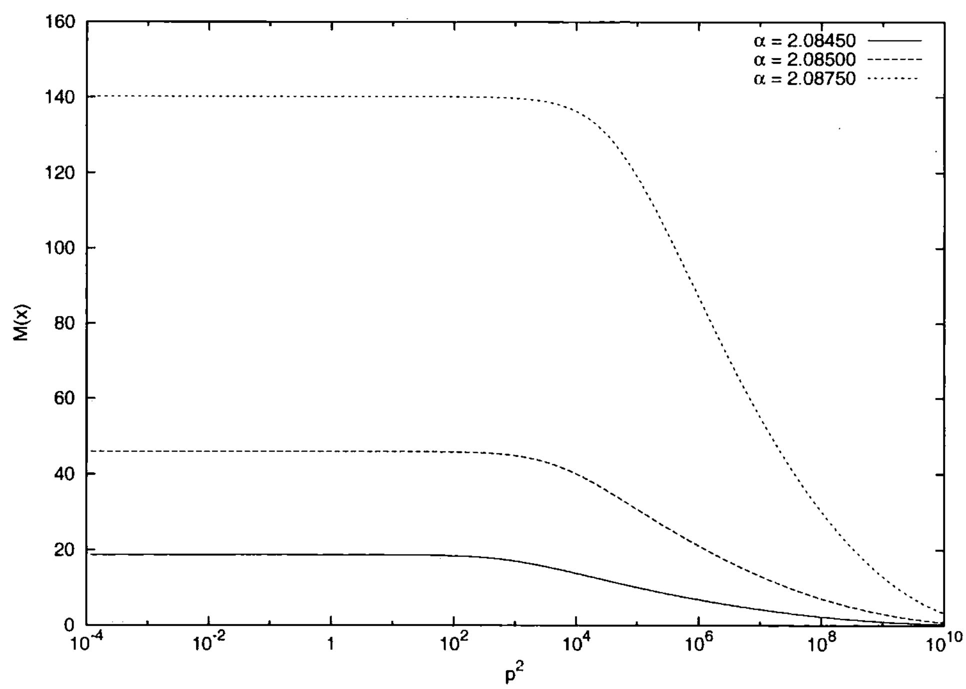

Figure 4.1: The Mass function $M(p^2, N_f = 1)$ renormalised at $\mu^2 = \Lambda^2$ for $\alpha_{1\ell}(\Lambda^2) = 2.08450, 2.08500, 2.08750$

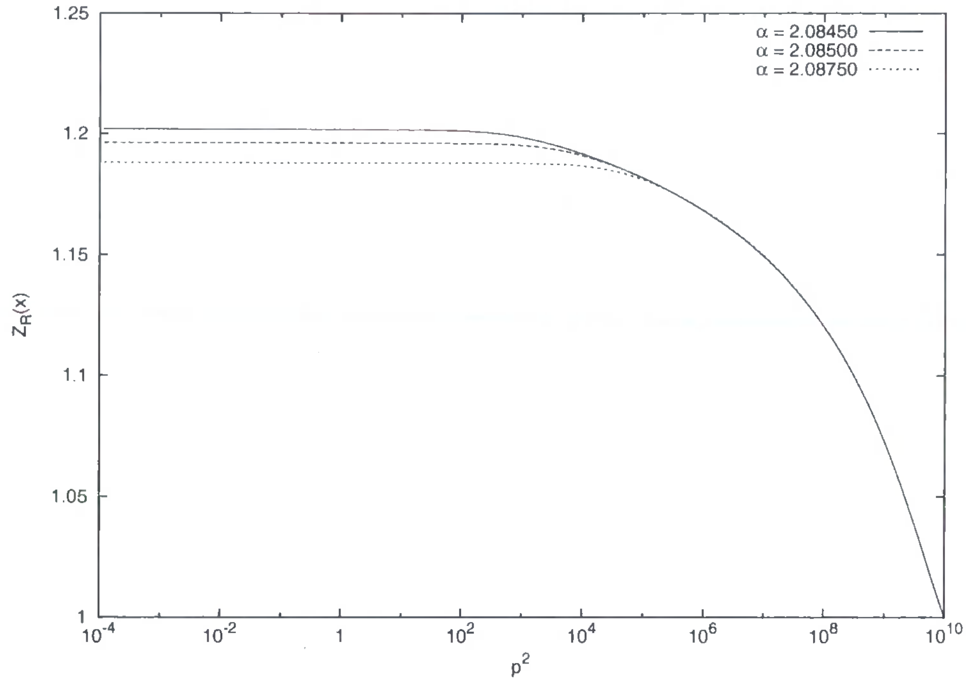

Figure 4.2: The dressing function $Z_R(p^2, \mu^2, N_f = 1)$ renormalised at $\mu^2 = \Lambda^2$ for $\alpha_{1\ell}(\Lambda^2) = 2.08450, 2.08500, 2.08750$

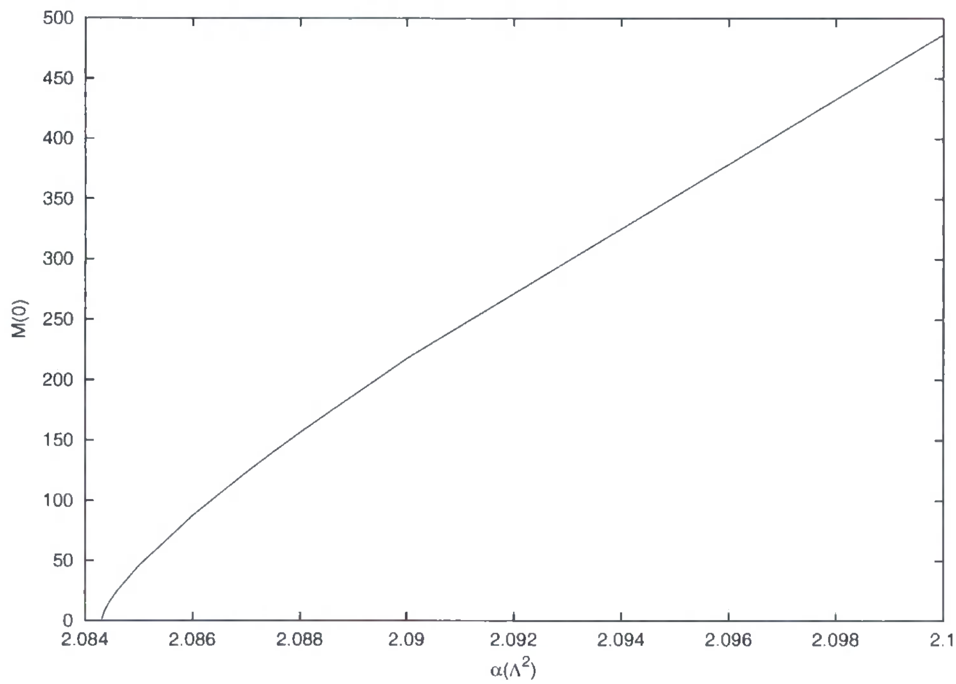

Figure 4.3: The infrared mass $M(0, N_f = 1)$ as a function of the bare coupling $\alpha_{1\ell}(\Lambda^2)$

In the bare vertex approximation, the non-renormalised system (M, Z) has been solved and the results were quoted in the first chapter. We recall that in this approximation, the critical coupling was

$$\alpha_{1\ell C} = 1.67280. \tag{4.78}$$

Compared to our value $\alpha_{1\ell C} = 2.084312$, we see that the renormalisation plus the introduction of a scheme that respects Multiplicative Renormalisability has increased the critical coupling by 25 %. Surprisingly, this value of the critical coupling Eq. (4.77) has been also found by Bloch in [3], where he solved the system $(M, \alpha_{1\ell}, Z \equiv 1)$ in the bare vertex approximation and where the integrals were regularised by a cut-off Λ^2.

We have also solved the system (M, Z) at one loop for a number of flavour $N_f = 2$. The critical coupling we find is

$$\alpha_{1\ell C}(\Lambda^2, N_f = 2) = 2.99142, \tag{4.79}$$

to be compared to [3]

$$\alpha_{1\ell C} = 2.02025, \tag{4.80}$$

in the bare vertex approximation of Bloch [3]. Here again, our value Eq. (4.79) is the same as the one found in [3] for the $(M, \alpha 1\ell, Z \equiv 1)$ in the bare vertex approximation and where the integrals were regularised by a cut-off Λ^2.

In Fig. (4.4) and Fig. (4.5) the typical behaviour for the mass function $M(p^2)$ and the dressing function $Z_R(p^2, \mu^2)$ renormalised at $\mu^2 = \Lambda^2$ for $N_f = 2$ is shown. We also plot in Fig. (4.6) the infrared mass $M(0, N_f = 2)$ as a function of the bare coupling $\alpha_{1\ell}(\Lambda^2)$.

In the next section we solve the full system (M, Z_R, α) and find out what happens to the critical coupling.

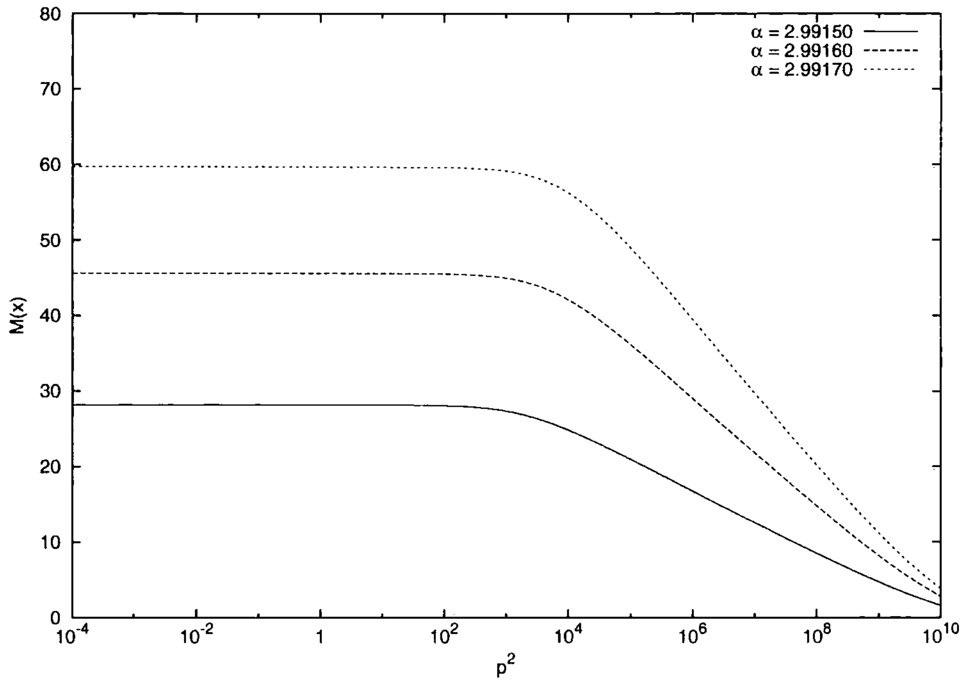

Figure 4.4: The Mass function $M(p^2, N_f = 2)$ renormalised at $\mu^2 = \Lambda^2$ for $\alpha_{1\ell}(\Lambda^2) = 2.99143, 2.99150, 2.99160$

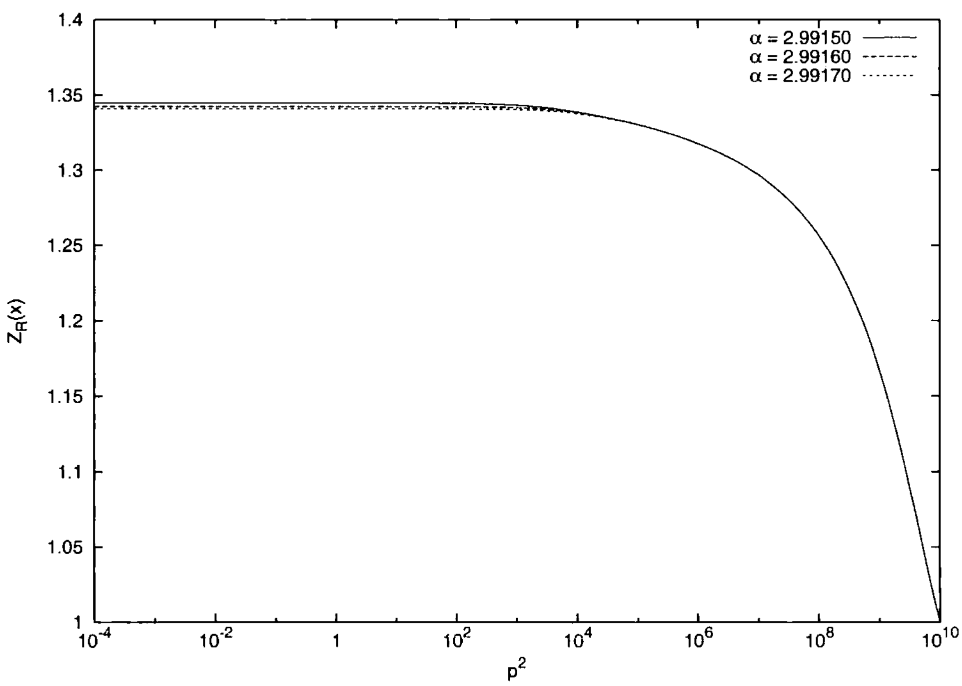

Figure 4.5: The dressing function $Z_R(p^2, \mu^2, N_f = 2)$ renormalised at $\mu^2 = \Lambda^2$ for $\alpha_{1\ell}(\Lambda^2) = 2.99143, 2.99150, 2.99160$

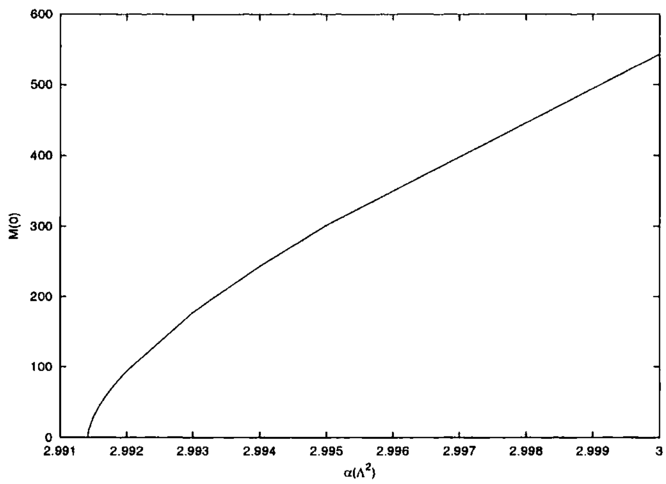

Figure 4.6: The infrared mass $M(0, N_f = 2)$as a function of the bare coupling $\alpha_{1\ell}(\Lambda^2)$

4.4 The System (M, Z_R, α) in the MR Scheme

For completeness we write the full system of integral equations involving the three functions $(M, Z_R$ and $\alpha)$

$$\frac{M(x)}{Z_R(x,\mu^2)} = M_\mu + \frac{3}{2\pi^2}\int dy\,\frac{M(y)}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)}\int d\theta\,\sin^2\theta\left[\frac{\alpha(z)}{z} - x \leftrightarrow \mu^2\right], \tag{4.81}$$

$$\begin{aligned}\frac{1}{Z_R(x,\mu^2)} &= 1 + \frac{1}{2\pi^2}\int dy\,\frac{1}{Z_R(y,\mu^2)}\frac{y}{y+M^2(y)} \\ &\times \int d\theta\,\sin^2\theta\left\{\alpha(z)\left[\frac{3\sqrt{y/x}\cos\theta}{z} - \frac{2y\sin^2\theta}{z^2}\right] - x \leftrightarrow \mu^2\right\},\end{aligned} \tag{4.82}$$

$$\begin{aligned}\frac{1}{\alpha(x)} &= \frac{1}{\alpha(\mu^2)} + \frac{4N_f}{3\pi^2}\int dy\frac{y}{y+\Sigma^2(y)} \\ &\times \int d\theta\left\{\frac{\sin^2\theta}{x(z+\Sigma^2(z))}\left[y(1-4\cos^2\theta) + 3\sqrt{yx}\cos\theta\,\right] - x \leftrightarrow \mu^2\right\}.\end{aligned} \tag{4.83}$$

As usual, the equation for α has been subtracted by its value at $x = \mu^2$ to remove the renormalisation factor $Z_3(\mu^2, \Lambda^2)$. The angular integrals of the equation for M and Z_R only involve the coupling function $\alpha(z)$ and we thus use the same extrapolation as the one we used in the last section, i.e.

$$\alpha(x) = \alpha(\epsilon^2), \qquad x < \epsilon^2, \tag{4.84}$$

$$\alpha(x) = \alpha(\Lambda^2), \qquad x > \Lambda^2. \tag{4.85}$$

The angular integral of the integral equations for the coupling $\alpha(x)$ involves the mass function $M(z)$. We thus also have to extrapolate the mass function outside its definition range. In order to keep the continuity of the mass function,which is vital to ensure proper numerical behaviour, we will use the following extrapolation that assumes a decay of the mass function for high momentum

$$M(x) = M(\epsilon^2), \qquad x < \epsilon^2, \tag{4.86}$$

$$M(x) = M(\Lambda^2)\frac{\Lambda^2}{x}, \qquad x > \Lambda^2. \tag{4.87}$$

In this way, the mass function is no longer smooth at $x = \Lambda^2$, i.e. the derivative is not continuous at $x = \Lambda^2$ but the function $M(x)$ retains its continuity. In practice, the mass function at $\Lambda^2 = 10^{10}$ is still not very small and the use of this extrapolation amounts to neglecting the mass function for high momentum, which causes an infrared instability for the coupling function. In order to avoid a rapid fall of the coupling function in the infrared, we have to be careful with the high momentum behaviour of the mass function. In practice we will just freeze the mass function to its value at Λ^2 for p^2 above the cut-off Λ^2.

The best way to solve this system is to apply the global Newton method, we have used for the one loop approximation. Unfortunately, the presence of the coupling function α in the angular integrals with argument z, which can go outside the range of momentum poses a problem to define the Jacobian, i.e. the derivative with respect to the expansion coefficients c_j of the coupling function α. This derivative is well defined for $x < \Lambda^2$, but is ill-defined when the argument of α is the variable z.

Moreover, the application of the Newton method will require a recalculation of the radial as well as the angular integrals at each iteration. If we choose 50 expansion coefficients, we will have to solve a 150×150 system at each iteration with the task of recomputing the coefficients of this system. This is why we say that the non-linear system is almost, but not quite algebraic. It would have been algebraic if the coefficients of the Jacobian equation $\mathbf{J}.\delta\mathbf{x} = \mathbf{F}$ had been constants. We thus have to look for another method to solve the full system, trying to keep the advantages of the Newton method. If we fix the coupling function $\alpha(x)$, then by using the global Newton method, we can solve the subsystem (M, Z_R) efficiently. The solutions $(M_{\text{new}}, Z_{R,\text{new}})$ can thus be used to recompute the new coupling function α_{new} using the integral equation Eq. (4.83). This new coupling function is reused to solve the subsystem (M, Z_R) to produce another approximation to the coupling function. We repeat this iteration until we achieve convergence. In order to implement this procedure, we thus need to expand the three functions M, Z_R and α on the Chebyshev basis using the t logarithmic momentum scale. We write

$$M(t) = \sum_{j=1}^{N_M}{}' a_j T_{j-1}(s), \tag{4.88}$$

$$Z(t) = \sum_{j=1}^{N_Z}{}' b_j T_{j-1}(s), \tag{4.89}$$

$$\alpha(t) = \sum_{j=1}^{N_\alpha}{}' c_j T_{j-1}(s) \tag{4.90}$$

with

$$s = \frac{t - \frac{1}{2}(t_{\text{max}} + t_{\text{min}})}{t_{\text{max}} - t_{\text{min}}}. \tag{4.91}$$

We introduce the Gauss-Legendre quadrature to rewrite the integrals as sums, i.e

$$\frac{M(t_i)}{Z_R(t_i)} = \frac{3}{2\pi^2}\sum_{j=1}^{N_R} w_j \frac{M(t_j)}{Z_R(t_j)}\frac{y_j^2}{y_j^2 + M^2(t_j)}\Theta_M(t_i, t_j), \qquad i = 1, \ldots, N_M, \tag{4.92}$$

$$\frac{1}{Z_R(t_i)} = 1 + \frac{1}{2\pi^2}\sum_{j=1}^{N_R} \frac{w_j y_j^2 Z_R(t_j)}{y_j^2 + M^2(t_j)}\left\{\frac{1}{x_i}\Theta_Z(t_i, t_j) - x \leftrightarrow \mu^2\right\}, \; i = 1, \ldots, N_Z, \tag{4.93}$$

$$\frac{1}{\alpha(t_i)} = \frac{1}{\alpha(\mu^2)} + \frac{4N_f}{3\pi^2}\sum_{j=1}^{N_\alpha}\frac{w_j y_j^2}{y_j+\Sigma^2(y)}\left\{\frac{1}{x_i}\Theta_\alpha(t_i,t_j) - x \leftrightarrow \mu^2\right\},\ i=1,\ldots,N_\alpha, \tag{4.94}$$

with the angular integrals Θ_M and Θ_Z of Eq. (4.62-4.63). The angular integral Θ_α is

$$\Theta_\alpha(t_i,t_j) = \sum_{k=1}^{N_\alpha} w_k \sin^2\theta_k \frac{1}{z_k + M^2(z_k)}\left[y_j\left(1-4\cos^2\theta_k\right) + 3\sqrt{x_i y_j}\cos\theta_k\right], \tag{4.95}$$

with $z_k = x_i + y_j - 2\sqrt{x_i y_j}\cos\theta_k$ and N_α the number of angular points for the Gauss-Legendre quadrature rule. We now have defined all the quantities we needed. We will thus start our procedure by choosing a starting guess α_0 for the coupling function $\alpha(x)$ that we choose to be the one loop expression Eq. (4.42). Using this one loop expression we solve the subsystem (M, Z_R), and use the solution to compute the new coupling function using its definition of Eq. (4.94). we write

$$\alpha(t_i) = \left[\frac{1}{\alpha(\mu^2)} + \frac{4N_f}{3\pi^2}\sum_{j=1}^{N_\alpha} w_j \frac{y_j^2}{y_j+\Sigma^2(y)}\left\{\frac{1}{x_i}\Theta_\alpha(t_i,t_j) - x \leftrightarrow \mu^2\right\}\right]^{-1}, \tag{4.96}$$

and use this expression to compute the new expansion coefficients c_j using the formula of Eq. (3.14). Once the new c_j are determined, we recompute the coupling function α using Clenshaw's recurrence formula and solve the subsystem (M, Z_R), with the extrapolated new coupling function. We stop the iteration when the expansion coefficients c_j have reached a relative accuracy of 10^{-4}. For $N_f = 1$ and $N_f = 2$, the critical couplings $\alpha_C(\Lambda^2)$ are the same as the ones in the one loop approximation. In order to compare the one loop approximation with the full QED case, we plot for $N_f = 1$ in Fig. (4.7) the mass function $M(p^2)$ for the full QED case as well as the mass function $M_{1\ell}(p^2)$ in the one loop approximation. We repeat the same procedure in Fig. (4.8) for the renormalisation function Z_R and in Fig. (4.9) for the coupling constant. Lastly, we plot in Fig. (4.10), the infrared mass $M(0)$ for full QED as well as its one loop expression for comparison.

The coupling constant $\alpha(q^2)$ develops a plateau as soon as $p^2 \leq 10^3$. We recall that the one loop expression for the coupling constant α_{1l} contains a term $\log(\Lambda^2/p^2)$

which shows that it has no infrared mass cut-off. The equation for the coupling constant $\alpha(q^2)$ involves two fermion propagators and therefore we should expect that it should behave according to $\log(p^2 + M^2)$, where M is the generated mass. When M is much bigger than the momentum p^2, it becomes a constant explaining the appearance of the plateau. If we look at Fig. (4.7), we see that the function $M(p^2)$ starts dropping around $p^2 = 10^3$, which is consistent with the fact that the integral equation for M involves a term $1/\left(y + M^2(y)\right)$. A look at the behaviour of the coupling function shows that the plateau occurs around the same value of p^2, i.e. when the mass function starts to dominate.

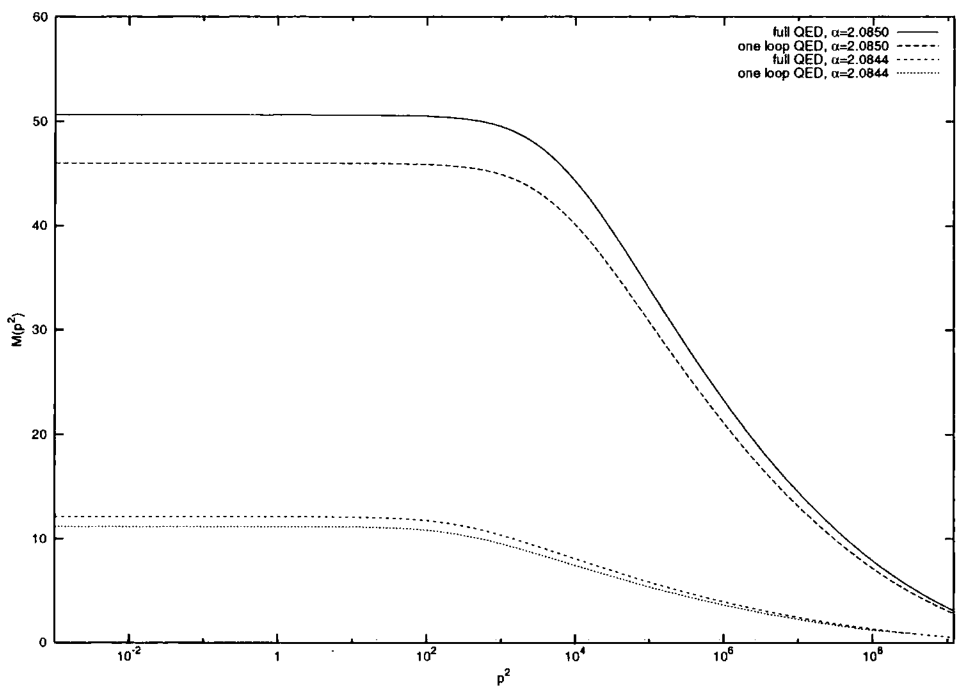

Figure 4.7: The Mass function $M(p^2, N_f = 1)$ renormalised at $\mu^2 = \Lambda^2$ for full QED and compared to its one loop expression at $\alpha(\Lambda^2) = 2.08440, 2.08500$

4.5 The Non-Local Gauge Fixing Method

In this section, we present a method to reduce the number of integral equations to be solved for the fermion propagator. The following material is presented to connect with other studies and therefore no numerical calculations will be attempted. The

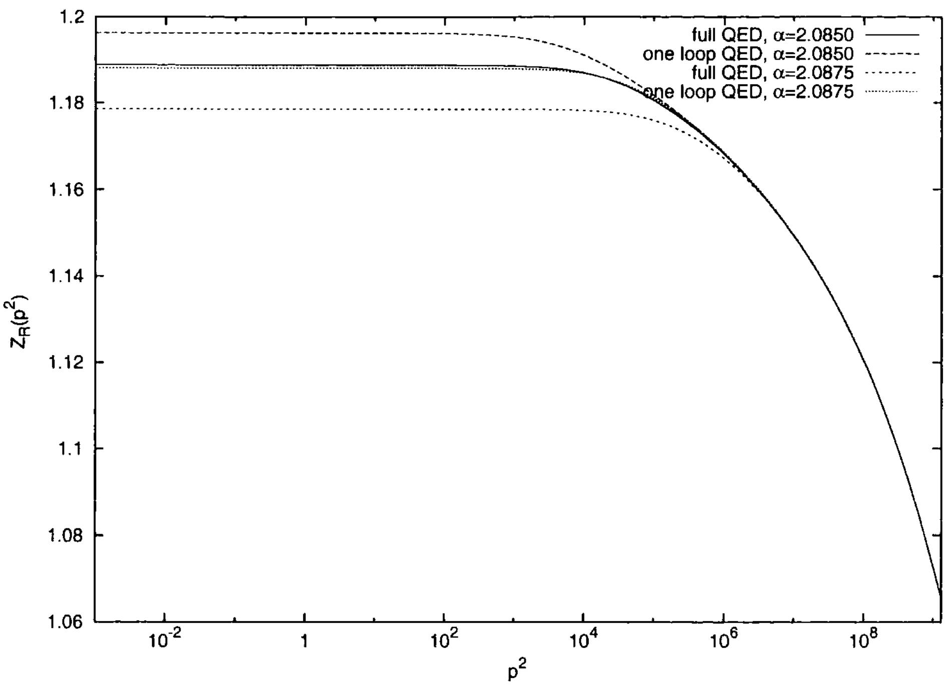

Figure 4.8: The dressing function $Z_R(p^2, \mu^2, N_f = 1)$ renormalised at $\mu^2 = \Lambda^2$ for full QED and compared to its one loop expression at $\alpha(\Lambda^2) = 2.08440, 2.08500$

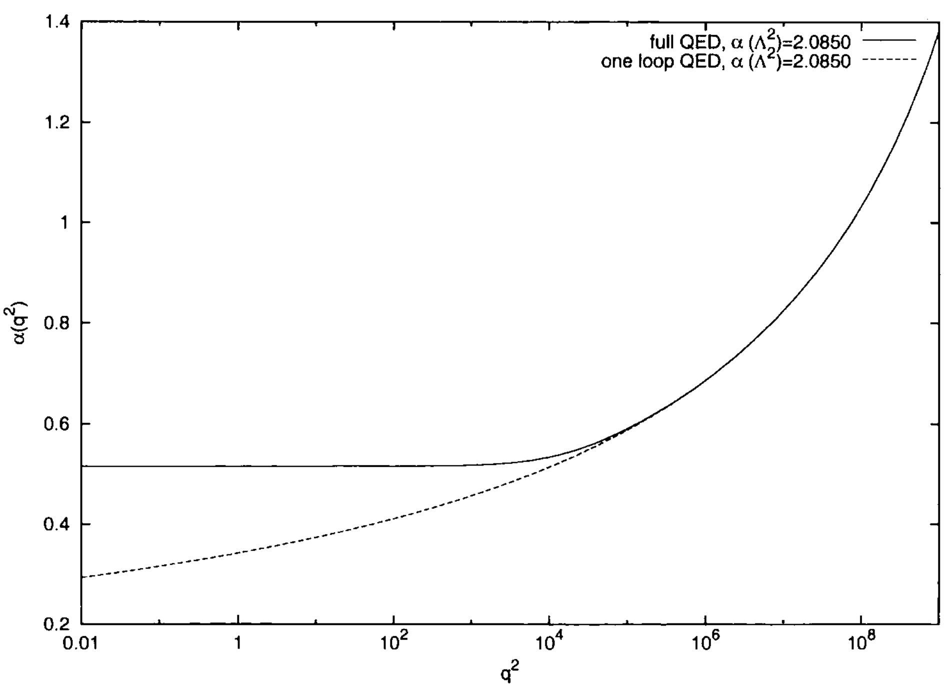

Figure 4.9: The coupling function $\alpha(q^2)$ for $N_f = 1$ and its one loop expression

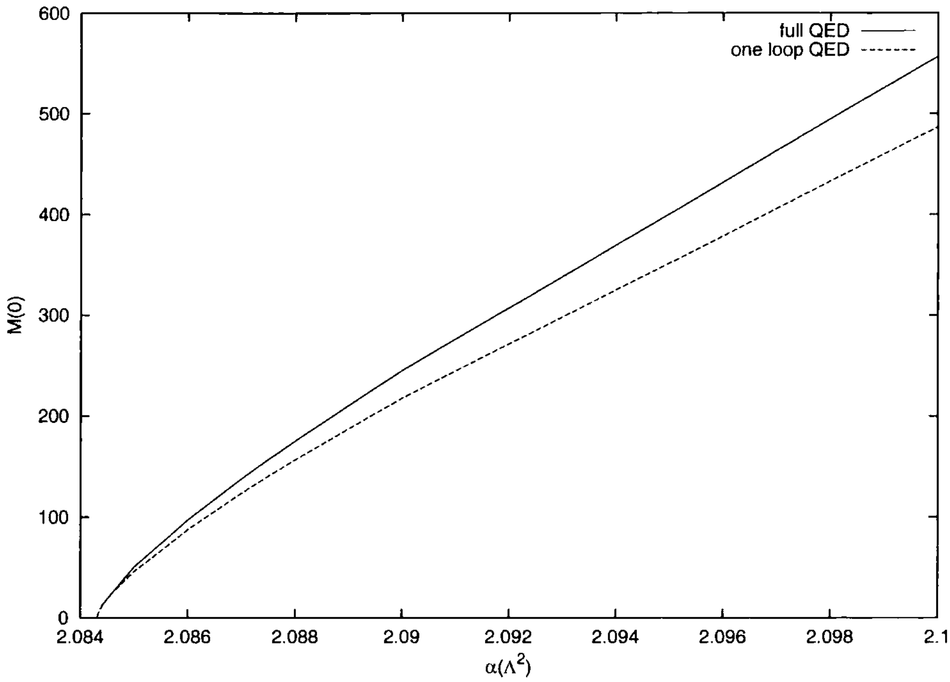

Figure 4.10: The infrared mass $M(0)$ for full QED as a function of $\alpha(\Lambda^2)$ for $N_f = 1$ and its one loop expression

non-local gauge fixing method is based on the observation that the gauge fixing parameter ξ_0, usually thought of as a number can be generalised to a function dependent on the momentum $z = (p-q)^2$, i.e.

$$\xi_0 = \xi_0(z). \tag{4.97}$$

It is then possible to choose the specific form of $\xi_0(z)$ so as to make the wave function factor $Z_R(p^2, \mu^2) \equiv Z_2(\mu^2, \Lambda^2)$, which is equivalent to setting the bare dressing function $Z(p^2, \Lambda^2)$ to unity, i.e. $Z(p^2, \Lambda^2) \equiv 1$.

In the bare vertex approximation the integral equations satisfied by $Z_R(p^2, \mu^2)$ and $M(p^2)$ are

$$\begin{aligned} \frac{1}{Z(x)} &= 1 - \frac{\alpha(\Lambda^2)}{2\pi^2 x} \int dy \, \frac{yZ(y)}{y + M^2(y)} \\ &\times \int d\theta \, \frac{\sin^2\theta}{z} \left\{ G(z) \left[\frac{2xy\sin^2\theta}{z} - 3\sqrt{yx}\cos\theta \right] \right. \\ &\left. + \, \xi \left[\frac{(y+x)\sqrt{yx}\cos\theta - 2yx}{z} \right] \right\} . \end{aligned} \tag{4.98}$$

$$\frac{M(x)}{Z(x)} = m_0 + \frac{\alpha(\Lambda^2)}{2\pi^2} \int dy \, \frac{yZ(y)M(y)}{y + M^2(y)} \int d\theta \, \frac{\sin^2\theta}{z} \Big\{ 3G(z) + \xi \Big\} . \tag{4.99}$$

(4.100)

In the MR scheme, we were able to write these equations so as to make the coupling function enter the angular integral. This was done in Landau gauge and was thus straightforward. If we make the gauge fixing parameter ξ_0 dependent on z in the following way

$$\xi_0 \equiv \xi_0(z) = \xi(z)G(z), \tag{4.101}$$

then we can factor out the photon dressing function $G(z)$ and rewrite the system as follows

$$\frac{M(x)}{Z(x)} = m_0 + \frac{1}{2\pi^2} \int dy \, \frac{yZ(y)M(y)}{y + M^2(y)} K_M(x,y) \, , \tag{4.102}$$

$$\frac{1}{Z(x)} = 1 - \frac{1}{2\pi^2 x} \int dy \, \frac{yZ(y)}{y + M^2(y)} K_Z(x,y) \, . \tag{4.103}$$

The integral kernel K_Z is given by

$$K_Z(x,y) = \int_0^\pi d\theta \, \sin^2\theta \, \alpha(z) \left[(3 - \xi(z)) \, \frac{\sqrt{xy}\cos\theta}{z} - 2\,(1 - \xi(z)) \, \frac{xy}{z^2} \sin^2\theta \right] , \tag{4.104}$$

which is obtained by using

$$x + y = z - 2\sqrt{xy}\cos\theta,$$

$$2xy = 2xy\cos^2\theta + 2xy\sin^2\theta.$$

We first note that

$$\sin^2\theta\cos\theta = \frac{1}{3}\frac{d}{d\theta}\left(\sin^3\theta\right) , \tag{4.105}$$

and rewrite K_Z as

$$\begin{aligned} K_Z(x,y) = & -\frac{1}{3}\int_0^\pi d\theta \, \sin^3\theta \sqrt{xy}\frac{d}{d\theta}\left([3 - \xi(z)] \, \alpha(z)\frac{1}{z} \right) \\ & -2\int_0^\pi d\theta \, \sin^4\theta \, [1 - \xi(z)] \, \alpha(z)\frac{xy}{z^2}, \end{aligned} \tag{4.106}$$

after integrating the cos term by parts.

The θ differentiation $d/d\theta$ is replaced by a differentiation with respect to the variable z since

$$\frac{d}{d\theta} = 2\sqrt{xy}\sin\theta\frac{d}{dz}\,, \tag{4.107}$$

which allows us to write

$$K_Z(x,y) = -2xy\int_0^\pi d\theta\,\sin^4\theta\left[\frac{1}{3}\frac{d}{dz}\left((3-\xi(z))\frac{\alpha(z)}{z}\right)+(1-\xi(z))\frac{\alpha(z)}{z^2}\right]. \tag{4.108}$$

The condition $K_Z \equiv 0$ can be guaranteed if $\xi(z)$ satisfies the differential equation

$$\frac{1}{3}\frac{d}{dz}\left((3-\xi(z))\frac{\alpha(z)}{z}\right)+(1-\xi(z))\frac{\alpha(z)}{z^2} = 0. \tag{4.109}$$

The solution of Eq. (4.109) is [15]

$$\xi(z) = \frac{3}{\alpha(z)z^2}\int_0^z dv v^2\frac{d}{dv}\alpha(v), \tag{4.110}$$

where the integration constant is taken so as to make $\xi(z)$ regular at $z=0$.

Using the non-local gauge fixing parameter Eq. (4.110), the dressing function $Z(x)$ can be set unity and we are left with only one equation for the mass function $M(x)$, i.e.

$$\frac{M(x)}{Z(x)} = m_0 + \frac{1}{2\pi^2}\int dy\,\frac{yZ(y)M(y)}{y+M^2(y)}K_M(x,y)\,, \tag{4.111}$$

with

$$K_M(x,y) = \int d\theta\,\frac{\sin^2\theta}{z}\alpha(z)\left[3+\xi(z)\right]\,, \tag{4.112}$$

where $\xi(z)$ is given by its representation Eq. (4.110).

The application of the non-local gauge technique to the MR scheme is straightforward. The equations are the same as the one in the ladder approximations, but with the amendments required to satisfy multiplicative renormalisation. The non-local gauge makes the dressing function $Z_R(x,\mu^2)$ equal the renormalisation constant $Z_2(\mu^2,\Lambda^2)$, which shows that $Z_R(x,\mu^2)$ is just a constant dependent on μ^2. The QED system in the non-local gauge will thus be

$$\frac{M(x)}{Z(x)} = \frac{1}{2\pi^2 x}\int dy\,M(y)\frac{y}{y+M^2(y)}\int d\theta\,\sin^2\theta\,\frac{\alpha(z)}{z}\left[3+\xi(z)\right], \tag{4.113}$$

$$\frac{1}{\alpha(x)} - \frac{1}{\alpha(\mu^2)} = \frac{4N_f}{3\pi^2} \int dy \frac{y}{y + M^2(y)} \tag{4.114}$$
$$\times \int d\theta \left\{ \frac{\sin^2\theta}{z + \Sigma^2(z)} \left[\frac{y}{x}\left(1 - 4\cos^2\theta\right) + 3\sqrt{\frac{y}{x}}\cos\theta \right] - x \leftrightarrow \mu^2 \right\},$$

$$\xi(z) = \frac{3}{\alpha(z)z^2} \int_0^z dv\, v^2 \frac{d}{dv}\alpha(v). \tag{4.115}$$

despite the interest of this method, we will not dwell on it here numerically since we are more interested in the prediction of the MR scheme for the quark equation in QCD, which is the topic of the next chapter.

Chapter 5

A New Truncation Scheme for the Quark Equation in QCD

We propose in this chapter a new truncation scheme for the quark equation in QCD. The truncation preserves Multiplicative Renormalisability as in the MR scheme and assumes a non-perturbative cancellation mechanism for the quark-gluon vertex that is different from but similar to the one in the MR scheme.

5.1 The Quark Equation

We start from the SD equation for the quark propagator in Euclidean QCD

$$[S_f(p)]^{-1} = \left[S_f^0(p)\right]^{-1} - C_F g_0(\Lambda^2) \int \frac{d^4q}{(2\pi)^4} \gamma_\mu S_f(q) \Gamma_\nu^{qg}(p, q, -r) D^{\mu\nu}(r), \tag{5.1}$$

where S_F and S_f^0 are the full and bare quark propagators respectively, Γ_ν^{qg} is the full quark-gluon vertex, C_F the colour factor with $C_F = 4/3$ for $N_c = 3$, g_0 the bare coupling constant and $r = p - q$.

The most general expressions for the full and bare quark propagators, the gluon and ghost propagators are

$$\begin{aligned} S_f^0(p) &= \frac{1}{i\, \not{p} + m_0}, \\ S_f(p) &= \frac{Z(p^2, \Lambda^2)}{i\, \not{p} + M(p^2)}, \end{aligned} \tag{5.2}$$

$$D_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\frac{F(p^2,\Lambda^2)}{p^2} + \xi_0 \frac{p^\mu p^\nu}{p^4}, \tag{5.3}$$

$$\Delta(p) = -\frac{G(p^2,\Lambda^2)}{p^2}. \tag{5.4}$$

We have made explicit the dependence on the cut-off Λ^2 to indicate possible divergences.

We use these expressions and derive the following relations for the mass function $M(p^2)$ and the dressing function $Z(p^2)$

$$\frac{1}{Z(p^2,\Lambda^2)} = 1 + \frac{C_F}{16\pi^4} g_0^2(\Lambda^2) \int d^4q\, Z(q^2,\Lambda^2) \frac{F(r^2,\Lambda^2)}{q^2 + M(q^2)} U_Z, \tag{5.5}$$

$$\frac{M(p^2)}{Z(p^2,\Lambda^2)} = m_0(\Lambda^2) - \frac{C_F}{16\pi^4} g_0^2(\Lambda^2) \int d^4q\, Z(q^2,\Lambda^2) M(q^2) \frac{F(r^2,\Lambda^2)}{q^2 + M(q^2)} U_M, \tag{5.6}$$

with the kernels U_Z and U_M both depending on the momenta p^2, q^2, r^2, Λ^2 and given by

$$\begin{aligned} U_Z(p^2,q^2,r^2,\Lambda^2) &= \frac{1}{4p^2r^2}\mathrm{Tr}\left[\not{p}\gamma_\mu\left(\not{q} + iM(q^2)\right)\Gamma_\nu^{qg}(q,p,-r,\Lambda^2)\right] \\ &\times \left[\delta_\perp^{\mu\nu}(r) + \frac{\xi}{F(r^2,\Lambda^2)}\frac{r^\mu r^\nu}{r^2}\right], \end{aligned} \tag{5.7}$$

$$\begin{aligned} U_M(p^2,q^2,r^2,\Lambda^2) &= \frac{1}{4r^2}\mathrm{Tr}\left[\gamma_\mu\left(1 - i\frac{\not{q}}{M(q^2)}\right)\right]\Gamma_\nu^{qg}(p,q,-r,\Lambda^2) \\ &\times \left[\delta_\perp^{\mu\nu}(r) + \frac{\xi}{F(r^2,\Lambda^2)}\frac{r^\mu r^\nu}{r^2}\right]. \end{aligned} \tag{5.8}$$

As usual, the Λ^2 dependence of the integrals is cancelled by the Λ^2 dependence of the bare mass $m_0(\Lambda^2)$, such that the mass function is finite and independent of Λ^2.

5.1.1 Renormalisation and truncation

These equations involve bare quantities only and are multiplicatively renormalised by writing

$$Z(p^2,\Lambda^2) = Z_2(\mu^2,\Lambda^2) Z_R(p^2,\mu^2), \tag{5.9}$$

$$
\begin{aligned}
F(p^2,\Lambda^2) &= Z_3(\mu^2,\Lambda^2)F_R(p^2,\mu^2), && (5.10)\\
G(p^2,\Lambda^2) &= \tilde{Z}_3(\mu^2,\Lambda^2)G_R(p^2,\mu^2), && (5.11)\\
g(\mu^2) &= \frac{Z_3^{1/2}(\mu^2,\Lambda^2)Z_2(\mu^2,\Lambda^2)}{Z_{1f}(\mu^2,\Lambda^2)}g_0(\Lambda^2), && (5.12)\\
g(\mu^2) &= \frac{Z_3^{1/2}(\mu^2,\Lambda^2)\tilde{Z}_3(\mu^2,\Lambda^2)}{\tilde{Z}_1(\mu^2,\Lambda^2)}g_0(\Lambda^2), && (5.13)\\
g(\mu^2) &= \frac{Z_3^{3/2}(\mu^2,\Lambda^2)}{Z_1(\mu^2,\Lambda^2)}g_0(\Lambda^2), && (5.14)
\end{aligned}
$$

where the dependence on the ultra-violet cut-off Λ^2 is traded by a dependence on the arbitrary momentum scale μ^2. Z_{1f}, Z_1 and $\tilde{Z}_1$ are the quark-gluon, triple gluon and ghost gluon vertex renormalisation constants respectively. Also Z_2, Z_3 and $\tilde{Z}_3$ are the quark, gluon and ghost field renormalisation constants respectively. Because of gauge invariance, the coupling is universal and can be written as the three different expressions Eq. (5.12-5.14). The renormalised equations for the quark propagator are thus

$$
\begin{aligned}
\frac{1}{Z_R(p^2)} &= Z_2 + \frac{C_F}{4\pi^3}\alpha(\mu^2)Z_{1f}^2\int d^4q\, Z_R(q^2)\frac{F_R(r^2)}{q^2+M(q^2)}U_Z, && (5.15)\\
\frac{M(p^2)}{Z_R(p^2)} &= m_0(\Lambda^2)Z_2 - \frac{C_F}{4\pi^3}\alpha(\mu^2)Z_{1f}^2\int d^4q\, Z_R(q^2)M(q^2)\frac{F_R(r^2)}{q^2+M(q^2)}U_M, && (5.16)
\end{aligned}
$$

where we have not indicated the dependence on the momenta scale μ^2 and Λ^2 explicitly. The subscript R denotes renormalised quantities and thus implies a μ^2 dependence. The renormalised functions $Z_{1f,2}$ are all $Z_{1f,2}(\mu^2,\Lambda^2)$. Because of the universality of the coupling, we can rewrite Z_{1f} as a function of Z_2

$$
Z_{1f} = \frac{Z_2}{\tilde{Z}_3}\tilde{Z}_1. \tag{5.17}
$$

This relation is the extension of the relation $Z_1 = Z_2$ that we have in QED. In QED, because of the equality of Z_1 and Z_2, we can factor out Z_2 and get rid of all renormalisation constants. We propose to do the same here and introduce the factor $\tilde{Z}_3^{-2}$ as well as a factor Z_2 in the integral by replacing them by their definition in

Eq. (5.9-5.11) and choose to apply it for the momentum r^2. The equation for M/Z_R becomes

$$\frac{M(p^2)}{Z_R(p^2)} = m_0 Z_2 - \frac{C_F}{4\pi^3} Z_2 \tilde{Z}_1^2 \alpha(\mu^2) \int d^4q\, M(q^2) \frac{F_R(r^2) G_R^2(r^2)}{q^2 + M(q^2)} \frac{Z(q^2)}{G^2(r^2)} U_M. \quad (5.18)$$

We now note as previously that the product

$$\hat{\alpha}(q^2) \equiv \alpha(\mu^2) \tilde{Z}_1^2(\mu^2, \Lambda^2) F_R(q^2, \mu^2) G_R^2(q^2, \mu^2) \quad (5.19)$$

is independent of μ^2 by rewriting it as

$$\hat{\alpha}(q^2) = \alpha(\Lambda^2) F(q^2, \Lambda^2) G^2(q^2, \Lambda^2). \quad (5.20)$$

We thus fix $\mu^2 = r^2$ and use the renormalisation conditions

$$F_R(q^2, q^2) = 1, \quad (5.21)$$

$$G_R(q^2, q^2) = 1, \quad (5.22)$$

to write the definite expressions

$$\frac{1}{Z_R(p^2)} = Z_2 + \frac{C_F}{4\pi^3} Z_2 \int d^4q \frac{\alpha(r^2)}{q^2 + M(q^2)} \left[\frac{Z(q^2, \Lambda^2)}{G^2(r^2, \Lambda^2)} U_Z(p^2, q^2, r^2, \Lambda^2) \right], \quad (5.23)$$

$$\frac{M(p^2)}{Z_R(p^2)} = m_0(\Lambda^2) Z_2 - \frac{C_F}{4\pi^3} Z_2 \int d^4q \frac{M(q^2)\alpha(r^2)}{q^2 + M(q^2)} \left[\frac{Z(q^2, \Lambda^2)}{G^2(r^2, \Lambda^2)} U_M(p^2, q^2, r^2, \Lambda^2) \right]. \quad (5.24)$$

Though up to now we have not introduced any approximation, it is clear that we will assume that the full quark-gluon vertex receives a $G^2(r^2)/Z(q^2)$ correction and make the replacement

$$\frac{Z(q^2, \Lambda^2)}{G^2(r^2, \Lambda^2)} U_M(p^2, q^2, r^2, \Lambda^2) \rightarrow U_M^0(p^2, q^2, r^2), \quad (5.25)$$

$$\frac{Z(q^2, \Lambda^2)}{G^2(r^2, \Lambda^2)} U_Z(p^2, q^2, r^2, \Lambda^2) \rightarrow U_Z^0(p^2, q^2, r^2), \quad (5.26)$$

where the subscript 0 indicates that we use bare quantities in the evaluation of the kernels $U_{M,Z}$.

We now introduce the notation

$$\begin{aligned}
\Sigma_Z(p^2) &= \frac{C_F}{4\pi^3}\int d^4q \, \frac{\alpha(r^2)}{q^2+M(q^2)}\left[\frac{Z(q^2,\Lambda^2)}{G^2(r^2,\Lambda^2)}U_Z(p^2,q^2,r^2,\Lambda^2)\right], \\
\Sigma_M(p^2) &= \frac{C_F}{4\pi^3}\int d^4q \, \frac{M(q^2)}{q^2+M(q^2)}\alpha(r^2)\left[\frac{Z(q^2,\Lambda^2)}{G^2(r^2,\Lambda^2)}U_M(p^2,q^2,r^2,\Lambda^2)\right],
\end{aligned} \tag{5.27}$$

and notice that neither Σ_Z nor Σ_M depend on the momentum scale μ^2 or on the renormalisation constant $Z_2(\mu^2,\Lambda^2)$. The only μ^2 dependence is through $Z_2(\mu^2,\Lambda^2)$ and it is straightforward to check that the system Eq. (5.23-5.24) preserves multiplicative renormalisation. By multiplying both equations in the system Eq. (5.23-5.24) by the factor $Z_R(\nu^2,\mu^2)$, it becomes clear after noting that

$$\begin{aligned}
Z_R(p^2,\nu^2) &= \frac{Z_R(p^2,\mu^2)}{Z_R(\nu^2,\mu^2)}, \\
Z_2(\nu^2,\Lambda^2) &= Z_R(\nu^2,\mu^2)Z_2(\mu^2,\Lambda^2),
\end{aligned}$$

that $Z_R(p^2,\nu^2)$ and $M(p^2)$ is a solution of the system if the pair $(Z_R(p^2,\mu^2), M(p^2))$ is a solution. We rewrite the system of equations in a more compact form as

$$\frac{1}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2)\left(1+\Sigma_Z(p^2)\right), \tag{5.28}$$

$$\frac{M(p^2)}{Z_R(p^2,\mu^2)} = Z_2(\mu^2,\Lambda^2)\left(m_0-\Sigma_M(p^2)\right), \tag{5.29}$$

which can, after elimination of the factor Z_2 from the first equation, be rearranged as

$$\frac{1}{Z_2(\mu^2,\Lambda^2)} = Z_R(p^2,\mu^2)\left(1+\Sigma_Z(p^2)\right), \tag{5.30}$$

$$M(p^2) = m_0-\left(M(p^2)\Sigma_Z(p^2)+\Sigma_M(p^2)\right). \tag{5.31}$$

If we look at Eq. (5.31), we can see that it does not depend in any way on μ^2. In other treatment, such as the MR scheme, the mass m_0 is multiplied by a factor $Z_2(\mu^2,\Lambda^2)$, which cancels the μ^2 dependence of the integral. In our scheme, we were able to get rid of the μ^2 dependence for the mass function $M(p^2)$ and only left an explicit dependence on $Z_2(\mu^2, \Lambda^2)$ through the equation for Z_R. To eliminate $Z_2(\mu^2, \Lambda^2)$ from Eq. (5.30), we proceed as usual by subtracting at the momentum $x = \mu^2$ and obtain

$$Z_R(p^2, \mu^2) = 1 - Z_R(p^2, \mu^2)\Sigma_Z(p^2) + \Sigma_Z(\mu^2), \tag{5.32}$$

$$M(p^2) = m_0 - \left(M(p^2)\Sigma_Z(p^2) + \Sigma_M(p^2)\right). \tag{5.33}$$

5.1.2 Model coupling

The system of equations Eq. (5.32-5.33) is the basis of study of chiral symmetry breaking in QCD, once we have specified the form of the coupling $\alpha(q^2)$. The gluon propagator satisfies its own SD equation and should be incorporated here to have a complete system. For the moment, this task is too ambitious and so we resort to a modelling of the coupling function. For the coupling function we use the following form, first introduced in [4]

$$\alpha(q^2) \equiv \alpha(t\Lambda^2{}_{QCD}) = \frac{1}{c_0 + t^2}\left[c_0\alpha_0 + \frac{4\pi}{\beta_0}\left(\frac{1}{\log t} - \frac{1}{t-1}\right)\right], \tag{5.34}$$

where $t = q^2/\Lambda^2_{\mathrm{QCD}}$and $\beta_0 = (11N_c - 2N_f)/3$. The first term in the square bracket is responsible for the infrared fixed point of the coupling function, which is an expected feature of the coupling function [16, 17, 18, 19] , and the second term reproduces the correct perturbative behaviour. The factor $1/(t-1)$ is introduced to subtract the simple pole at $q^2 = \Lambda^2_{\mathrm{QCD}}$ [21] to make the coupling analytic for all spacelike momenta. If we use the coupling function which has no pole in perturbation theory (PT), we obtain the so called analytical perturbation theory (APT), which converges more rapidly than the usual perturbation theory [22] . We will fix the value of α_0 to 2.6, which is believed to be a good approximation of the IR fixed point. The value c_0 is for the moment fixed to 15 and $N_f = 1$. In a coming section treating the infrared behaviour of the gluon and ghost propagators, we will give more details about how to find the values of the parameter α_0. The coupling function with these parameters is shown in Fig. (5.1) with the one loop expression to show the cross over from perturbative behaviour to IR fixed point

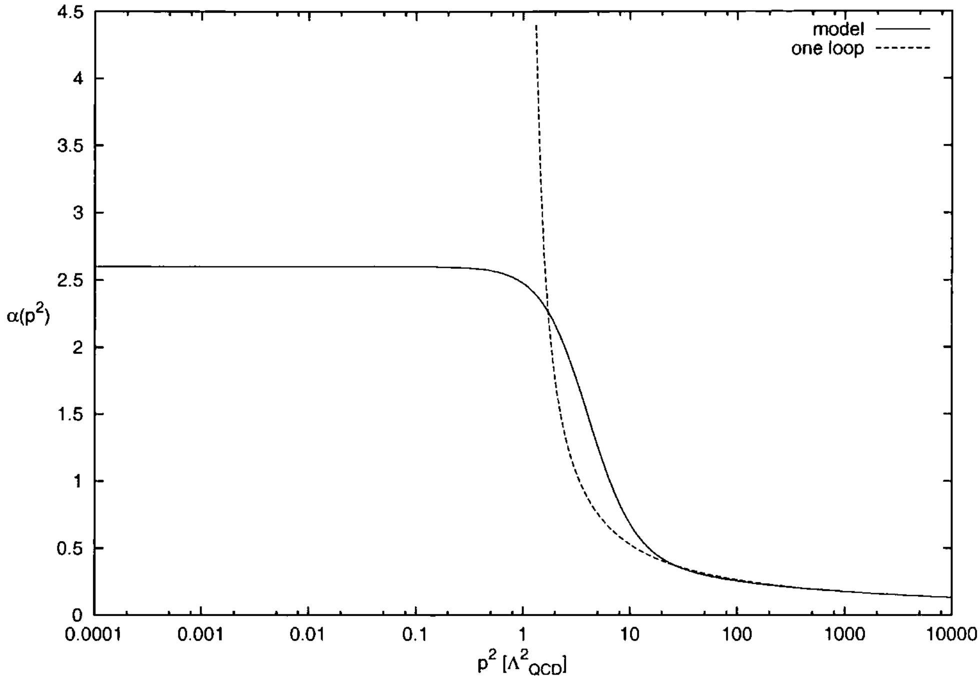

Figure 5.1: The model coupling function $\alpha(q^2)$ in units of $\Lambda^2_{\rm QCD}$ with $\alpha_0 = 2.6$ and $c_0 = 15$

As we have already noted neither $\Sigma_M(p^2)$ nor $\Sigma_Z(p^2)$ depends on the wave function renormalisation $Z_R(p^2, \mu^2)$, so in effect the system of integral equations is just an equation for the mass function $M(p^2)$ alone

$$M(p^2)\left[1 + \Sigma_Z(p^2)\right] = m_0 - \Sigma_M(p^2)\,. \tag{5.35}$$

Once we have solved Eq. (5.35), we can compute the wave function renormalisation $Z_R(p^2, \mu^2)$ using the non-subtracted equation

$$\frac{1}{Z_R(p^2, \mu^2)} = Z_2(\mu^2, \Lambda^2)\left[1 + \Sigma_Z(p^2)\right]\,. \tag{5.36}$$

The normalisation factor $Z_2(\mu^2, \Lambda^2)$ will be removed by fixing

$$Z_R(\mu^2, \mu^2) = 1\,. \tag{5.37}$$

5.1.3 Chiral case

We first solve the equation Eq. (5.35) in the chiral case for the parameters we have quoted so as to be able to compare with the results of J.Bloch [4], whose truncation scheme assumes a G^2/Z^2 cancellation mechanism for the full quark-gluon vertex. The evolution of $M(x)$ and $Z_R(x,\mu^2)$ for $10^{-4} \leq x \leq 10^4$ in units of $\Lambda^2_{\rm QCD}$ is shown in Fig. (5.2) and Fig. (5.3).

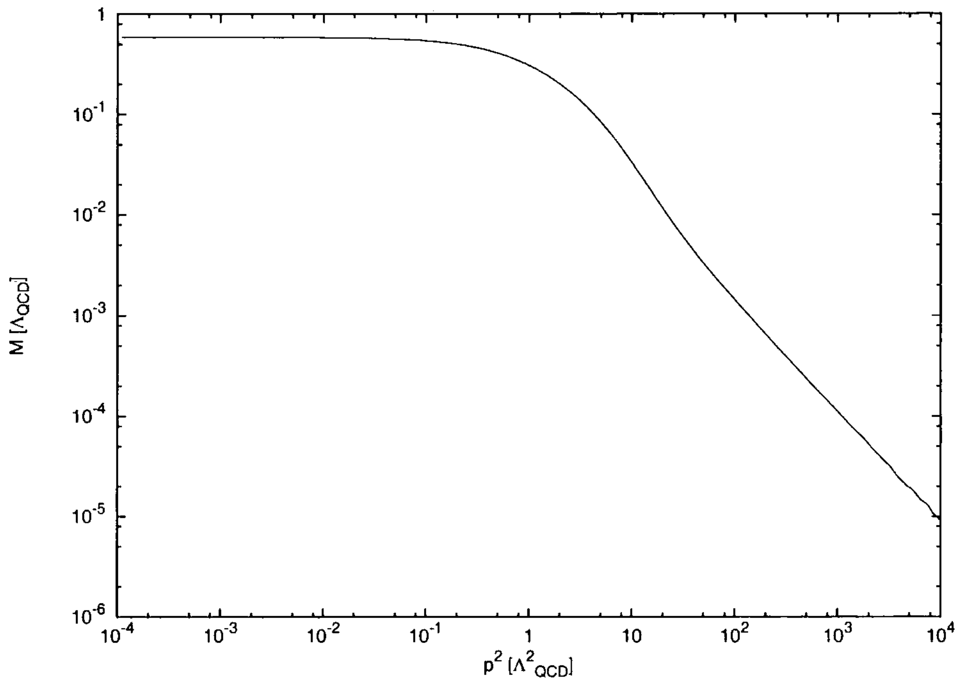

Figure 5.2: The mass function $M(p^2)$ in unit of $\mu^2 = \Lambda^2_{\rm QCD}$ with $\alpha_0 = 2.6$ and $c_0 = 15$

The wave function renormalisation $Z_R(x,\mu^2)$ is renormalised at $\Lambda^2_{\rm QCD}$. For these values of the parameters c_0 and α_0, we obtain a non-vanishing IR mass

$$M(0) = 0.575\Lambda_{\rm QCD}, \tag{5.38}$$

which is to be compared to

$$M_{\rm B}(0) = 1.057\Lambda_{\rm QCD}, \tag{5.39}$$

in the calculation of [4], where the generated mass $M_{\rm B}(0)$ is of the order of the extension of the infrared plateau of the mass function which happens to be the same

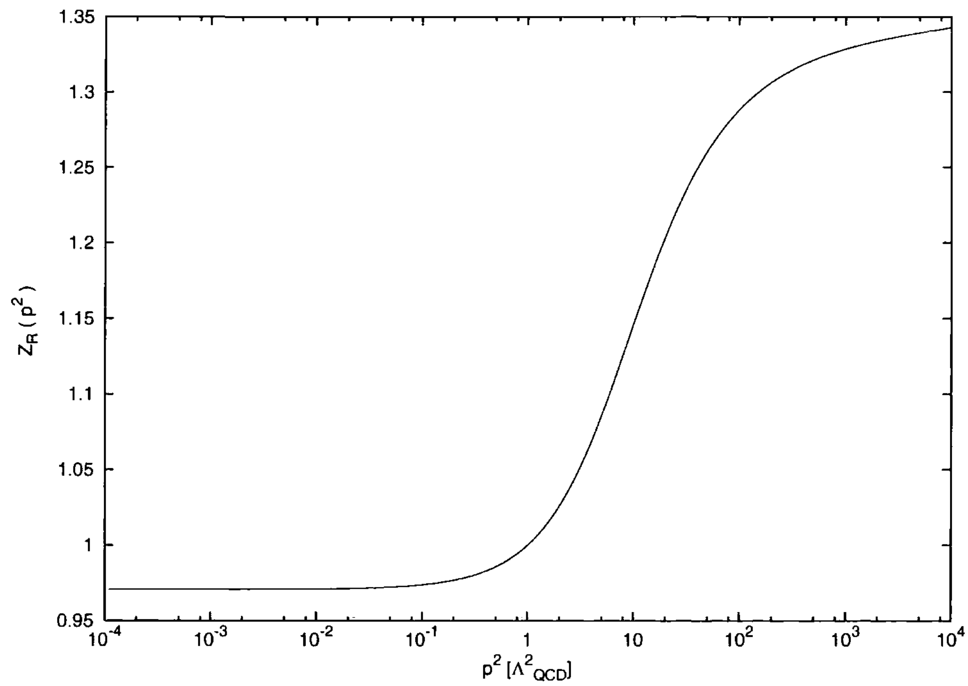

Figure 5.3: The wave function renormalisation $Z_R(p^2, \mu^2)$ renormalised at $\mu^2 = \Lambda^2_{\mathrm{QCD}}$

as the plateau of the coupling function $\alpha(q^2)$. In our case, the generated mass is around half of the extension of the infrared plateau of the coupling $\alpha(q^2)$ but is the same as the plateau of the mass function . Our infrared mass is about half the one calculated in [4] and therefore the cancellation mechanism assumed for the full quark-gluon vertex is relevant to the infrared behaviour of the mass function $M(p^2)$. In real QCD, we do not know which cancellation mechanism occurs or if any at all really occurs and more studies in this direction are needed. As already mentioned previously, it has been shown by Mandelstam [10], that perturbative loop corrections to the propagator introduce a factor G^2/Z^2. This cancellation mechanism is different from the one we assumed but it was shown to hold in perturbation theory. Nonperturbatively, the cancellation mechanism might be more subtle.

5.1.4 Sensitivity to α_0 and c_0

In this section, we study the sensitivity of the infrared mass $M(0)$ to the two parameters α_0 and c_0. The parameter c_0 fixes the behaviour of the coupling function

$\alpha(q^2)$ at intermediate momenta as shown in Fig. (5.4). In theory, this behaviour can only be known once we have solved the system of SD equations satisfied by the gluon, ghost and quark propagators.

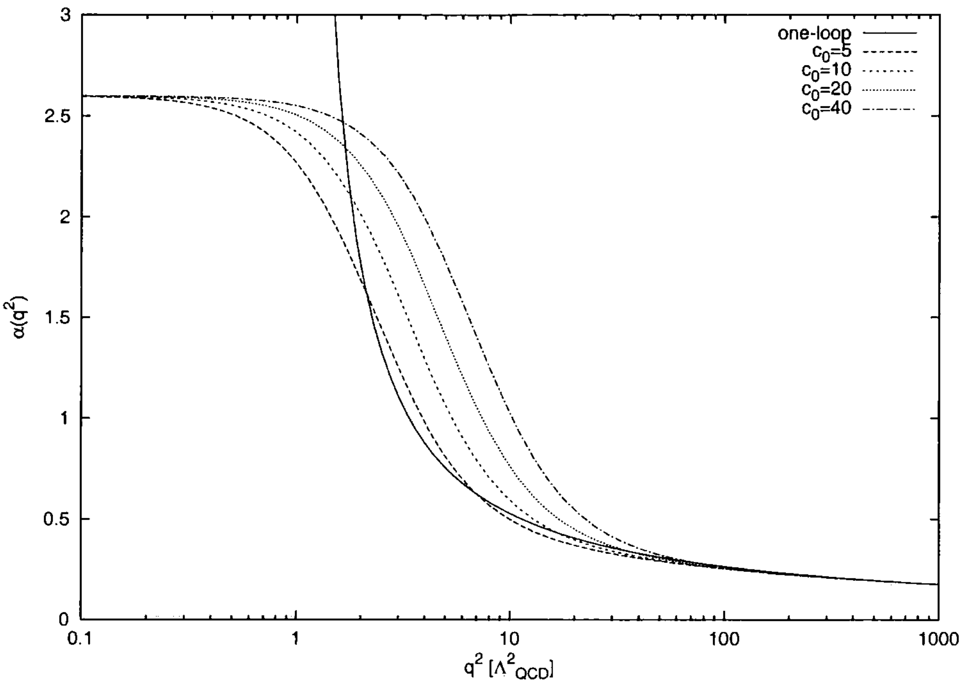

Figure 5.4: The coupling function $\alpha(q^2)$ for different c_0's with $\alpha_0 = 2.6$

In table 5.1, we give the value of $M(0)$ for different values of c_0 and compared to the calculation of [4], where as we have already mentioned the author assumed a G^2/Z^2 cancellation mechanism for the quark-gluon vertex. His equation for the mass function $M_B(p^2)$ even though independent of μ^2 involves the scale μ^2 unlike our scheme.

c_0	$M_B(0)$ [4]	M(0)
5	0.824	0.444
10	0.964	0.522
15	1.057	0.575
20	1.129	0.615
40	1.325	0.725

Table 5.1: The infrared mass $M(0)$ in units of Λ_{QCD} with $\alpha_0 = 2.6$ for different c_0 and compared to [4]

We now fix $c_0 = 15$ and plot in Fig. (5.5) the infrared mass $M(0)$ as a function of the infrared fixed point α_0. In table 5.2, we show some values and compare them to [4]. In our truncation scheme, the critical value α_0^{crit} for the appearance of an infrared mass is

$$\alpha_0^{\text{crit}} \approx 1.1, \tag{5.40}$$

whereas the value quoted in [4] is $\alpha_0^{\text{crit}} \approx 0.9$. For the same value of α_0, we are closer to our critical value and therefore it is normal to find a smaller infrared mass $M(0)$.

α_0	$M_B(0)$ [4]	M(0)
1	0.028	0.000
1.8	0.548	0.257
2.6	1.057	0.575
3.4	1.494	0.848
4.2	1.874	1.086

Table 5.2: The infrared mass $M(0)$ in units of Λ_{QCD} with $c_0 = 15$ for different α_0 and compared to [4]

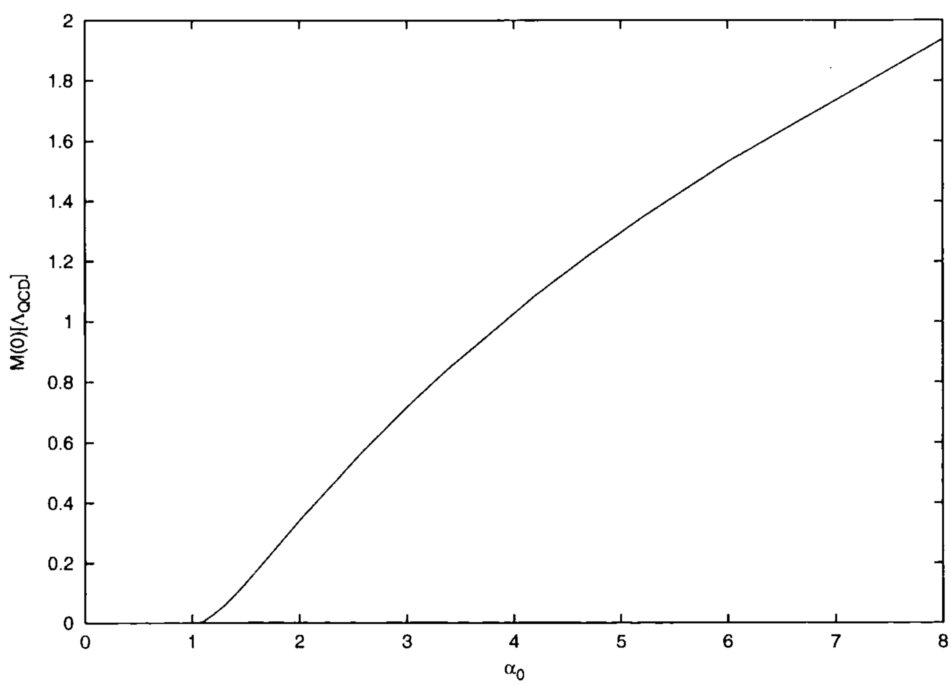

Figure 5.5: The infrared mass $M(0)$ in units of Λ_{QCD} as a function of α_0

5.1.5 Massive case

We now extend our study to the massive case. We just have to fix the bare mass m_0 to a non-zero value. We study the case $m_0 = 0.0001, 0.001, 0.01, 0.1$ in units of Λ_{QCD} with $N_f = 1$ and where $Z_R(p^2)$ is renormalised at $\mu^2 = 10^5$ and the case $m_0 = 1$ in units of Λ_{QCD}, with $N_f = 1, 3$ because for such an ultraviolet mass it is sensible to consider that we have more than one flavour propagating in the loop correction to the gluon propagator.

We plot in Fig. (5.6) and Fig. (5.7), the behaviour of the mass function $M(p^2)$ and $Z_R(p^2)$ respectively for $m_0 = 0.0001, 0.001, 0.01, 0.1$ and in Fig. (5.8) and Fig. (5.9) for $m_0 = 1$.

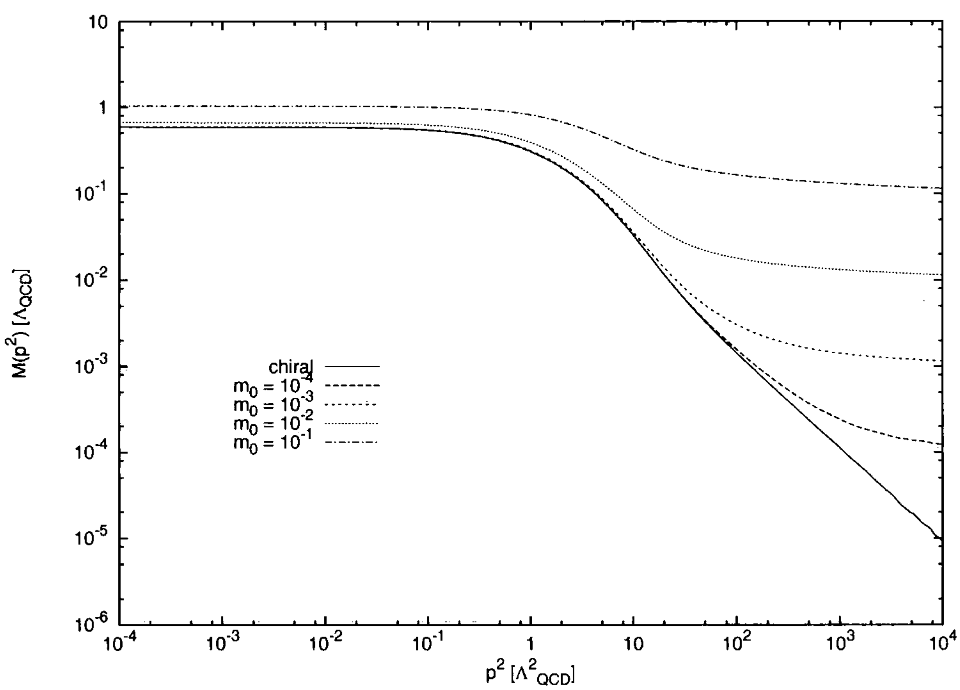

Figure 5.6: The mass function $M(p^2, N_f = 1)$ for non zero bare mass $m_0 = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$

5.2 The Gluon-Ghost Sector

In this section, we apply the previous method to the gluon-ghost sector of QCD. We recall that our goal is to factor out the renormalisation functions Z_3 and $\tilde{Z}_3$ of

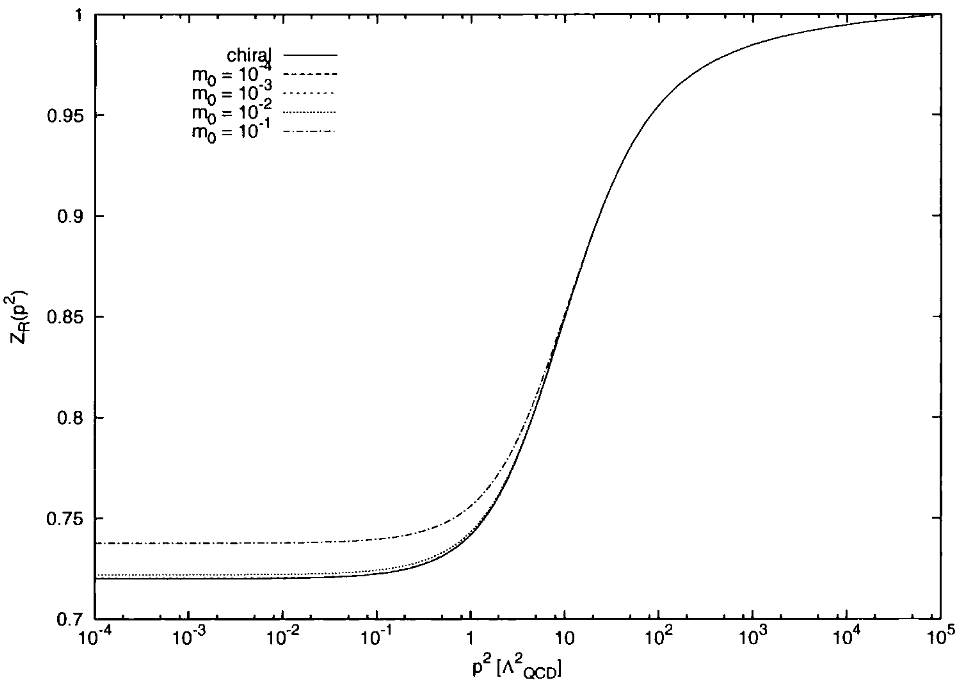

Figure 5.7: The renormalisation function $Z_R(p^2, N_f = 1)$ for non zero bare mass $m_0 = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ renormalised at $\mu^2 = 10^5$

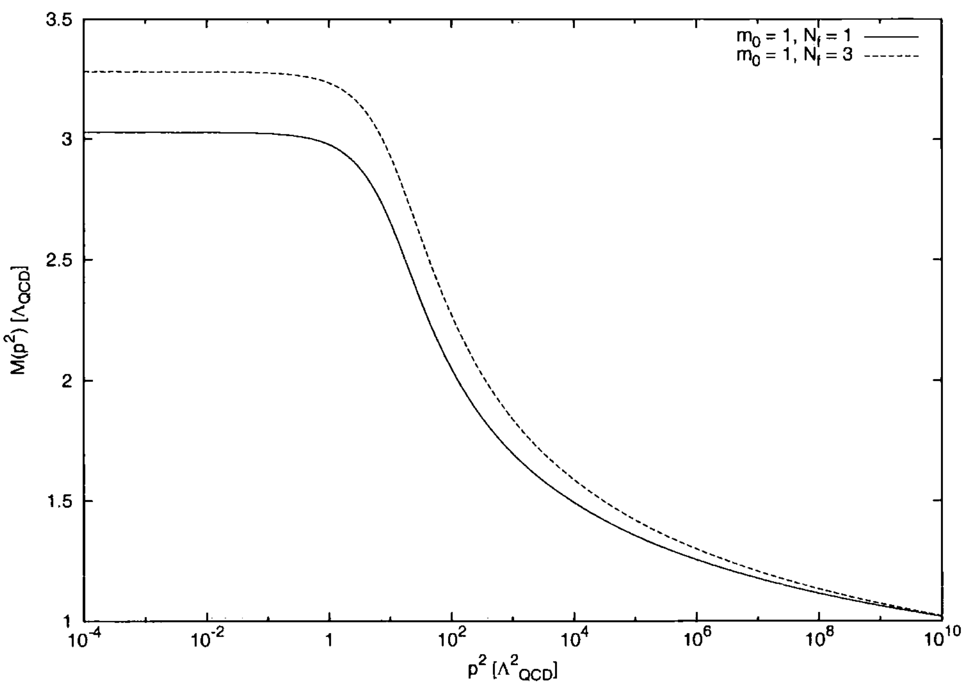

Figure 5.8: The mass function $M(p^2, N_f = 1, 3)$ for non zero bare mass $m_0 = 1$

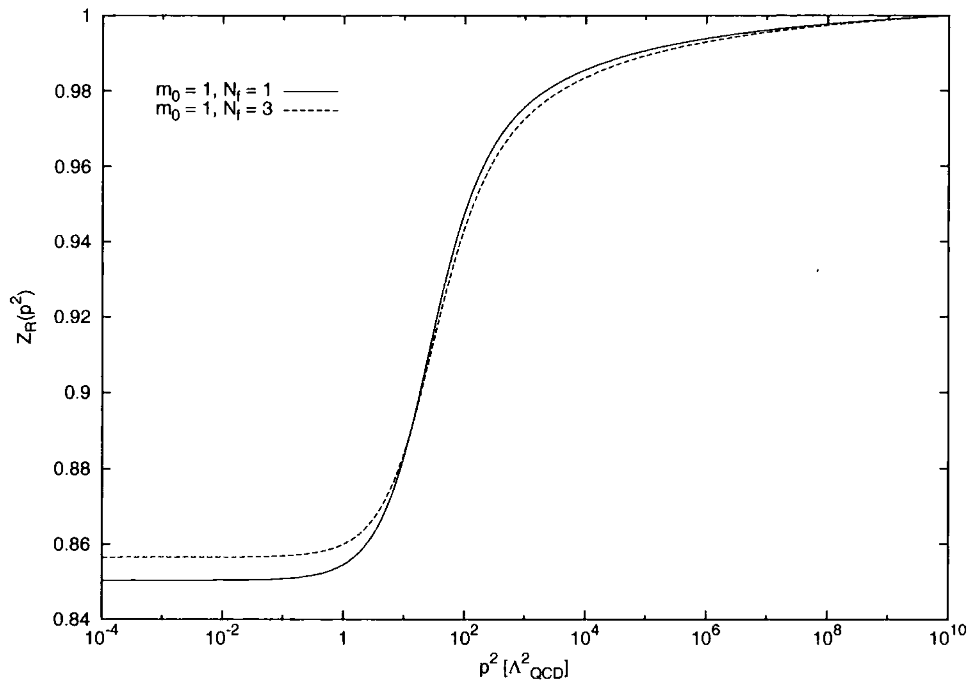

Figure 5.9: The renormalisation function $Z_R(p^2, N_f = 1, 3)$ for non zero bare mass $m_0 = 1$ renormalised at $\mu^2 = 10^{10}$

the gluon and ghost propagators, respectively. At the same time, we have seen that the running coupling function $\alpha(q^2)$ appears inside the integrals after eliminating the $\alpha(\mu^2)$ term that arises when we renormalise the coupling function. We shall do the same here and move the running coupling function $\alpha(q^2)$ inside the integral. The only μ^2 dependence left will be through the renormalisation functions Z_3, $\tilde{Z}_3$ and the dressing functions $F_R(p^2, \mu^2)$ and $G_R(p^2, \mu^2)$ of the gluon and ghost propagators, respectively.

5.2.1 The equations

In Minkowskian formulation, the *QCD* Schwinger-Dyson equations for the gluon-ghost sector, neglecting the quark contribution and four-gluon vertex, are

$$[\Delta(p)]^{-1} = \left[\Delta^0(p)\right]^{-1} - N_c g_0^2 \int \frac{d^4q}{(2\pi)^4} G^0_\mu(p,q)\Delta(q)G_\nu(q,p)D^{\mu\nu}(r)\,, \tag{5.41}$$

$$
\begin{aligned}
\left[D_{\mu\nu}(p)\right]^{-1} &= \left[D^0_{\mu\nu}(p)\right]^{-1} \\
&-(-1)N_c g_0^2 \int \frac{d^4q}{(2\pi)^4} G^0_\mu(-r,q)\Delta(q)G_\nu(q,-r)\Delta(-r) \\
&-\frac{1}{2}N_c g_0^2 \int \frac{d^4q}{(2\pi)^4} \Gamma^{3g,0}_{\mu\alpha_1\alpha_2}(-p,q,r)D^{\alpha_1\beta_1}(q)\Gamma^{3g}_{\nu\beta_1\beta_2}(p,-r,-q)D^{\alpha_2\beta_2}(r)\,,
\end{aligned}
\tag{5.42}
$$

where g_0 is the bare coupling, $D_{\mu\nu}$ the gluon propagator, Δ the ghost propagator, $\Gamma^{3g}_{\alpha\beta\gamma}$ the triple-gluon vertex, G_μ the ghost-gluon vertex, the superscript 0 indicating bare quantities and $r = p - q$.

The general expressions for the full gluon and ghost propagators in a covariant gauge ξ can be written as

$$
D_{\mu\nu}(p) = -i\left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{F(p^2)}{p^2} + \xi\frac{p_\mu p_\nu}{p^4}\right], \tag{5.43}
$$

$$
\Delta(p) = \frac{i\,G(p^2)}{p^2}, \tag{5.44}
$$

which is the Minkowski representation of Eq. (5.3-5.4).

As in QED, we introduce the projector $\mathcal{P}_{\mu\nu}$ defined as

$$
\mathcal{P}_{\mu\nu} = g_{\mu\nu} - 4\frac{p^\mu p^\nu}{p^2}, \tag{5.45}
$$

to avoid spurious quadratic divergence. In order to find equations for $F(p^2)$ and $G(p^2)$, we apply the projector $\mathcal{P}_{\mu\nu}$ to the gluon equation Eq. (5.41) and we multiply the ghost equation Eq. (5.41) by the factor i/p^2. After performing a Wick rotation and identifying $x = p^2, y = q^2, z = r^2$, we obtain [34]

$$
\begin{aligned}
\frac{1}{F(x)} &= 1 - \frac{N_c g_0^2}{8\pi^3}\int_0^{\Lambda^2}\int_0^\pi d\theta \sin^2\theta \; y dy \; \left[M(x,y,z)\,G(y)G(z)\right. \\
&\qquad\left. + Q(x,y,z)\,F(y)F(z)\right],
\end{aligned}
\tag{5.46}
$$

$$
\frac{1}{G(x)} = 1 - \frac{N_c g_0^2}{8\pi^3}\int_0^{\Lambda^2}\int_0^\pi d\theta \sin^2\theta \; y dy \; T(x,y,z)\,G(y)F(z)\,. \tag{5.47}
$$

The three kernels M, Q, T depends on the full and bare triple-gluon and ghost-gluon vertices and are given by [34]

$$
M(p^2,q^2,r^2) = \frac{1}{3p^2\,q^2\,r^2}\,\mathcal{P}^{\mu\nu}(p)\,G^0_\mu(-r,q)\,G_\nu(q,-r)\,, \tag{5.48}
$$

$$Q(p^2,q^2,r^2) = -\frac{1}{6p^2q^2r^2}\,\mathcal{P}^{\mu\nu}(p)\,\Gamma^{3g,0}_{\mu\alpha_1\alpha_2}(-p,q,r)\,\Gamma^{3g}_{\nu\beta_2\beta_1}(p,-r,-q) \tag{5.49}$$
$$\times\left[g_\perp^{\alpha_1\beta_1}(q)\,g_\perp^{\alpha_2\beta_2}(r)+\xi\left(g_\perp^{\alpha_1\beta_1}(q)\frac{r^{\alpha_2}r^{\beta_2}}{r^2F(r^2)}+g_\perp^{\alpha_2\beta_2}(r)\frac{q^{\alpha_1}q^{\beta_1}}{q^2F(q^2)}\right)\right.$$
$$\left.+\xi^2\frac{q^{\alpha_1}q^{\beta_1}}{q^2F(q^2)}\frac{r^{\alpha_2}r^{\beta_2}}{r^2F(r^2)}\right],$$

$$T(p^2,q^2,r^2) = -\frac{1}{p^2\,q^2\,r^2}\left[g_\perp^{\mu\nu}(r)+\xi\frac{r^\mu r^\nu}{r^2F(r^2)}\right]G^0_\mu(p,q)\,G_\nu(q,p)\,, \tag{5.50}$$

with $g_\perp^{\mu\nu}(q)=g^{\mu\nu}-q^\mu q^\nu/q^2$.

We now introduce the renormalised dressing functions F_R and G_R as well as the running coupling function $\alpha(\mu^2)$ from Eq. (5.10-5.12) to obtain [34]

$$\frac{1}{G_R(x)} = \tilde{Z}_3(\mu^2,\Lambda^2)-\frac{N_c\,\alpha(\mu^2)}{2\pi^2}\int_0^{\Lambda^2}\int_0^\pi d\theta\sin^2\theta\; ydy\;\, T(x,y,z)\,G_R(y)F_R(z)\,, \tag{5.51}$$

$$\frac{1}{F_R(x)} = Z_3(\mu^2,\Lambda^2)-\frac{N_c\alpha(\mu^2)}{2\pi^2}\int_0^{\Lambda^2}\int_0^\pi d\theta\sin^2\theta\; ydy \tag{5.52}$$
$$\times\left[\tilde{Z}_1^2(\mu^2,\Lambda^2)M(x,y,z)G_R(y)G_R(z)+Z_1^2(\mu^2,\Lambda^2)Q(x,y,z)F_R(y)F_R(z)\right].$$

We first consider the ghost equation Eq. (5.51). As we wish to introduce the running coupling $\alpha(p^2)$ in the integrand, we now make use of the μ^2 independent function $\hat{\alpha}(q^2)$ of Eq. (5.20) and obtain

$$\frac{1}{G_R(x)} = \tilde{Z}_3(\mu^2,\Lambda^2)-\frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^\pi d\theta\sin^2\theta\; ydy\;\,\alpha(z)T(x,y,z)\,\frac{G_R(y)}{G_R^2(z)}\,. \tag{5.53}$$

We now use $G=\tilde{Z}_3G_R$ and obtain

$$\frac{1}{G_R(x)} = \tilde{Z}_3(\mu^2,\Lambda^2)\left[1-\frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^\pi d\theta\sin^2\theta\; ydy\;\frac{G(y)}{G^2(z)}T(x,y,z)\,\alpha(z)\right]. \tag{5.54}$$

To treat the gluon equation Eq. (5.52), we first use

$$Z_1(\mu^2,\Lambda^2)=\frac{Z_3(\mu^2,\Lambda^2)}{\tilde{Z}_3(\mu^2,\Lambda^2)}\tilde{Z}_1(\mu^2,\Lambda^2)\,, \tag{5.55}$$

which can be derived from the universality relations Eq. (5.13-5.14). We also choose to introduce the coupling function $\alpha(z)$ in the integrand as we did for the ghost equation. After introducing bare dressing functions F and G through their definitions, we obtain

$$\frac{1}{F_R(x)} = Z_3(\mu^2,\Lambda^2)\left[1-\frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^{\pi} d\theta \sin^2\theta \, ydy \right.$$
$$\left. \times \tilde{Z}_1^2\left(\frac{G(y)}{G(z)F(z)}M(x,y,z)+\frac{F(y)}{G^2(z)}Q(x,y,z)\right)\alpha(z)\right] . \quad (5.56)$$

From the two equations Eq. (5.56-5.54), it is clear we will assume the following non-perturbative cancellation mechanism

$$\frac{G(q^2,\Lambda^2)}{G^2(r^2,\Lambda^2)}T(p^2,q^2,r^2,\Lambda^2) \rightarrow T^0(p^2,q^2,r^2)\,, \quad (5.57)$$

$$\frac{F(q^2,\Lambda^2)}{G^2(r^2,\Lambda^2)}Q(p^2,q^2,r^2,\Lambda^2) \rightarrow Q^0(p^2,q^2,r^2)\,, \quad (5.58)$$

$$\frac{G(q^2,\Lambda^2)}{G(r^2,\Lambda^2)F(r^2,\Lambda^2)}M(p^2,q^2,r^2,\Lambda^2) \rightarrow M^0(p^2,q^2,r^2)\,, \quad (5.59)$$

which amounts to assume that the ghost gluon vertex $G_\nu(q,p)$ receives a non-perturbative $G^2(r^2)/G(q^2)$ correction (in the kernel T) and that the triple-gluon vertex $\Gamma^{3g}_{\alpha\beta\gamma}(p,-r,-q)$ receives a nonperturbative $G^2(r^2)/F(q^2)$ correction (in the kernel Q) and that the ghost vertex $G_\nu(q,-r)$ receives a nonperturbative $G(r^2)F(r^2)/G(q^2)$ correction (in the kernel M). We have kept the dependence on the momenta p,q,r explicit since it is not obvious how the vertex gets corrected. We recall that in the treatment of the quark equation we assumed that the quark gluon vertex $\Gamma^{qg}_\nu(p,q,-r)$ received a non-perturbative $G^2(r^2)/Z(q^2)$, which is consistent with the cancellation mechanism we assume for the kernels T and Q. The kernel M has a different cancellation mechanism that could be attributed to the way it depends on the different momenta p,q,r.

The system of equations we obtain, after being able to introduce the running coupling inside the integrals and factoring out the renormalisation function Z_3 and $\tilde{Z}_3$, is (in Landau gauge)

$$\frac{1}{G_R(x)} = \tilde{Z}_3(\mu^2,\Lambda^2)\left[1-\frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^{\pi} d\theta\sin^2\theta\, ydy\, T^0(x,y,z)\,\alpha(z)\right]\,, \quad (5.60)$$

$$\frac{1}{F_R(x)} = Z_3(\mu^2,\Lambda^2)\Bigg[1 - \frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^{\pi} d\theta \sin^2\theta \, ydy \, \alpha(z) \times \Big(M^0(x,y,z) + Q^0(x,y,z)\Big)\Bigg], \quad (5.61)$$

with [34]

$$\begin{aligned} R_0(x,y,z) = M^0 + Q^0 &= -\frac{1}{3}\Bigg[\frac{x^2}{8\,y^2\,z^2} + \frac{x}{y\,z}\left(\frac{1}{y}+\frac{1}{z}\right) - \frac{1}{8}\left(\frac{15}{y^2}+\frac{34}{y\,z}+\frac{15}{z^2}\right) \\ &+\frac{1}{4\,x}\left(\frac{z}{y^2} - \frac{11}{y} - \frac{11}{z} + \frac{y}{z^2}\right) \\ &+\frac{1}{2\,x^2}\left(\frac{z^2}{y^2}+\frac{6\,z}{y} - 14 + \frac{6\,y}{z}+\frac{y^2}{z^2}\right)\Bigg], \end{aligned} \quad (5.62)$$

$$\begin{aligned} T_0(x,y,z) &= -\left(\frac{x}{y} - 2 + \frac{y}{x}\right)\frac{1}{4\,z^2} + \left(\frac{1}{y}+\frac{1}{x}\right)\frac{1}{2\,z} - \frac{1}{4\,x\,y} \\ &= \frac{\sin^2\theta}{z^2}\,. \end{aligned} \quad (5.63)$$

We can now write an equation that involves the running coupling $\alpha(x)$ only by using its definition in Landau gauge

$$\alpha(x) = \alpha(\mu^2) F_R(x,\mu^2) G_R^2(x,\mu^2)\,. \quad (5.64)$$

We have

$$\frac{1}{\alpha(x)} = \frac{Z_3(\mu^2,\Lambda^2)\,\tilde{Z}_3^2(\mu^2,\Lambda^2)}{\alpha(\mu^2)}\,\Sigma_F(x)\,\left[\Sigma_G(x)\right]^2, \quad (5.65)$$

where

$$\Sigma_G(x) = 1 - \frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^{\pi} d\theta \sin^2\theta \, ydy \, T^0(x,y,z)\,\alpha(z)\,, \quad (5.66)$$

$$\Sigma_F(x) = 1 - \frac{N_c}{2\pi^2}\int_0^{\Lambda^2}\int_0^{\pi} d\theta \sin^2\theta \, ydy \, \alpha(z) \times \Big(M^0(x,y,z) + Q^0(x,y,z)\Big)\,. \quad (5.67)$$

We now use the definition of the renormalised coupling Eq. (5.13) and obtain in Landau gauge

$$\frac{1}{\alpha(x)} = \frac{1}{\alpha(\Lambda^2)}\,\Sigma_F(x)\,\left[\Sigma_G(x)\right]^2, \quad (5.68)$$

which is a non-linear integral equation for the renormalised running coupling $\alpha(x)$ and where the renormalised constants Z_3 and $\tilde{Z}_3$ as well as any μ^2 dependence have been eliminated. It is interesting to note that the equation satisfied by the mass function $M(p^2)$ Eq. (5.35)depends on the functions $F_R(p^2)$ and $G_R(p^2)$ only through the coupling function $\alpha(q^2)$. Once we determine $\alpha(q^2)$, then the mass function $M(p^2)$ will be derived from it and $Z_R(p^2)$ will follow subsequently from Eq. (5.36). It would be interesting to see if this procedure can be applied to the full QCD case where we should include the two loops diagrams and the quark contribution to the polarisation tensor of the gluon.

5.2.2 Infrared behaviour

In this section we review how to find the infrared behaviour of the dressing functions for the gluon and ghost. We first write the equations satisfied by the gluon and ghost propagators. They are

$$\begin{aligned} [D_{\mu\nu}(p)]^{-1} &= \left[D^0_{\mu\nu}(p)\right]^{-1} - \pi^{gh}_{\mu\nu}(p) - \pi^{gl}_{\mu\nu}(p) \\ &\quad -\pi^{3g}_{\mu\nu}(p) - \pi^{4g}_{\mu\nu}(p) - \pi^{tad}_{\mu\nu}(p) - \pi^{q}_{\mu\nu}(p)\,, \end{aligned} \tag{5.69}$$

$$[\Delta(p)]^{-1} = \left[\Delta^0(p)\right]^{-1} - N_c g_0^2 \int \frac{d^4q}{(2\pi)^4}\, G^0_\mu(p,q)\Delta(q)G_\nu(q,p)D^{\mu\nu}(r)\,, \tag{5.70}$$

where the vacuum polarisation includes contributions from the ghost loop $\pi^{gh}_{\mu\nu}(p)$, gluon loop $\pi^{gl}_{\mu\nu}(p)$, three-gluon diagram $\pi^{3g}_{\mu\nu}(p)$, four-gluon diagram $\pi^{4g}_{\mu\nu}(p)$, tadpole diagram $\pi^{tad}_{\mu\nu}(p)$ and quark loop $\pi^{q}_{\mu\nu}(p)$. If the contribution from the quarks and two loop diagram is neglected, we arrive at a system that has been solved in several approximations. The simplest used was the angular approximation [17], where it was found that the behaviour of the dressing functions in the infrared was

$$F_R(x) \sim x^{2\kappa} \quad , \qquad G_R(x) \sim x^{-\kappa}\,, \tag{5.71}$$

These power laws lead to an infrared fixed point for the coupling function

$$\alpha_0 = \lim_{x\to 0} \alpha(\Lambda^2)\, F_R(x)\, G_R^2(x) \to \text{constant}\,. \tag{5.72}$$

In the Landau gauge, the bare vertex approximation of Eq. (5.70) for the renormalised ghost dressing function becomes,

$$\frac{1}{G_R(x,\mu^2)} = \tilde{Z}_3(\mu^2,\Lambda^2) - \frac{N_c\,\alpha_\mu}{4\pi^3}\int d^4q\, T_0(x,y,z)\, G_R(y,\mu^2)\, F_R(z,\mu^2)\,. \tag{5.73}$$

After substitution of the power laws (5.71) in Eq. (5.73), the right hand side yields a sum of integrals of the form

$$\int \frac{d^4q}{(2\pi)^4}\, x^\alpha\, y^\beta\, z^\gamma\,, \tag{5.74}$$

with $\alpha+\beta+\gamma=\kappa-2$. Integrals of this type can be easily calculated by introducing Feynman parameters, yielding [26, 27]

$$I(a,b) = \int \frac{d^4q}{(2\pi)^4}\,\frac{1}{y^a z^b} = \frac{1}{16\pi^2}\,\frac{\Gamma(2-a)\Gamma(2-b)\Gamma(a+b-2)}{\Gamma(a)\Gamma(b)\Gamma(4-a-b)}\, x^{2-a-b}\,. \tag{5.75}$$

Both sides of Eq. (5.73) yield a leading infrared power x^κ, and equating their coefficients gives

$$\alpha_0 = \chi_{\rm gh}(\kappa) = \frac{2\pi}{3N_c}\,\frac{\Gamma(3-2\kappa)\Gamma(3+\kappa)\Gamma(1+\kappa)}{\Gamma^2(2-\kappa)\Gamma(2\kappa)}\,. \tag{5.76}$$

The expression (5.76) gives the relation between the infrared fixed point α_0 and the exponent κ. If we use the gluon equation we will obtain another expression that would also relate α_0 and κ i.e.

$$\alpha_0 = \chi_{\rm gl}(\kappa)\,, \tag{5.77}$$

which, for consistency reason, has to be equal to $\chi_{\rm gh}(\kappa)$. This equality fixes the value of κ. Recent calculations [16, 17, 18] predicted $0.4 \le \kappa \le 1.0$ and lattice calculations [19, 20] predict $\kappa \approx 0.5$. For $\kappa = 0.5$ we obtain from the ghost equation $\alpha_0 = 5\pi/6 \approx 2.6$, i.e. the value that we used in the previous sections. The value $c_0 = 15$ was chosen so as to match the intermediate region to the results of [17].

We have seen that a study of the gluon-ghost sector at one loop predicts an infrared fixed point for the strong coupling. Obviously, it remains to be shown that this pattern subsists when one treats the gluon-ghost sector fully when it is coupled to the quark equation. In such a treatment, we will need a truncation and the one proposed in [4] or ours seem to be good candidate.

Chapter 6

Bound States Masses

6.1 Introduction

The masses of hadrons built from quarks and gluons are determined by the strong physics aspects of QCD. These can be calculated, for instance, by Lattice Monte Carlo methods. However, these simulations are only tractable if the lattice is not too big. This in turn means that the quark masses should not be too small. Crucially the hadron world delivered by nature has very light *up* and *down* current quark masses. To reach such small masses lattice results have to be extrapolated from larger values. This introduces a major uncertainty in the lattice "prediction"for the masses of light flavour hadrons. Rather than use some purely statistical method of extrapolation, such as spline fitting, it is clearly far more reliable to use a model that contains the correct physics. The NJL model [28, 29] is one that naturally embodies chiral symmetry breaking and is known to reproduce the physics of the pion. Within an $SU(2)$ flavour model one can in turn calculate the mass of the ρ and the strength of its interactions. It is known that this accords well with experiment for small quark masses.

The purpose of this chapter is to use the $SU(2)$ NJL model to calculate the masses of the π and ρ as a function of quark mass. These can then be compared with the dependence given by lattice computations in the region of overlap. If these agree, then the NJL model can serve as a reliable way of extrapolating to physical quark masses. In the second part of the chapter, we introduce a new technique to compute the effective action by controlling the quantum fluctuations by a parameter in the Lagrangian. We apply this method to a four fermion interaction where the control parameter is the mass of the fermion and to the bosonised form of the four fermion theory. The equations we obtain deserve further study to show if this scheme is an improvement over the usual NJL model.

6.2 THE $SU(2)$ NJL Model

Many models have been designed to understand the low energy sector of QCD. The approach to model building is that it should be simple enough, yet able to capture the main characteristics of the fundamental theory under scrutiny. The NJL model is such an attempt and it aims at a unified description of the physical vacuum of mesons and baryons. It makes use of an attractive interaction in the scalar $\bar{q}q$ channel, which is strong enough to cause chiral symmetry breaking and which gives quarks a constituent mass.

6.2.1 The Lagrangian

The Nambu Jona-Lasinio model describes a system of quarks with four fermion interactions. The two-flavour version of the model (up and down quarks) is defined by the Lagrangian

$$L_{NJL} = \bar{\psi}(x)(i\gamma^{\mu}.\partial_{\mu} - m_0) + L_{int} \tag{6.1}$$

In writing the Lagrangian we have assumed that the bare quark masses are degenerate ($m_u = m_d = m_0$). The four-fermion interaction is given by

$$\begin{aligned} L_{int} &= \frac{G_1}{2}\left[\left(\bar{\psi}(x)\psi(x)\right)^2 + \left(\bar{\psi}(x)i\gamma^5\tau^a\psi(x)\right)^2\right] + \\ &\quad \frac{G_2}{2}\left[\left(\bar{\psi}(x)\gamma_{\mu}\tau^a\psi(x)\right)^2 + \left(\bar{\psi}(x)i\gamma_{\mu}\gamma^5\tau^a\psi(x)\right)^2\right] \end{aligned} \tag{6.2}$$

The different four-fermion interactions will be referred to as the scalar, pseudoscalar, vector and axial interactions respectively. The dynamical fields are $\bar{\psi}$ and ψ. The two couplings G_1 and G_2 have the dimension M^{-2} or unit GeV^{-2}. G_1 is chosen positive and G_2 negative so as to make the interaction attractive in the quark-antiquark channels. As we are working with the $SU(2)$ formulation, the τ^a are just the three Pauli matrices.

6.2.2 Symmetries of the NJL model

Flavour symmetry

Under a flavour rotation the quark fields transform as follows

$$\psi \to \exp^{-i\theta_a \tau_a/2} \psi\,, \qquad \bar{\psi} \to \bar{\psi} \exp^{i\theta_a \tau_a/2}\,, \tag{6.3}$$

where the parameters θ_a are the amplitudes of the flavour rotation. In the limit $m_u = m_d$ where the masses are equal, the Lagrangian is invariant. This invariance with respect to isospin rotations gives rise to near degenerate isospin multiplets whose degeneracy is lifted by a few MeV due to the small difference between the u and d masses. In QCD, the invariance with respect to flavour rotations stems from the fact that the quark-gluon interaction is flavour independent and flavour rotation is broken by the flavour dependence of the current quark masses.

Chiral symmetry

A chiral transformation acts on the quark fields as follows

$$\psi \to \exp^{-i\gamma^5 \theta_a \tau_a/2} \psi\,, \qquad \bar{\psi} \to \bar{\psi} \exp^{-i\gamma^5 \theta_a \tau_a/2}\,. \tag{6.4}$$

In the chiral limit, where $m_u, m_d \to 0$, the Lagrangian is formally invariant, but this symmetry is spontaneously broken, and the vacuum is not invariant. The appearance of massless Goldstone bosons is the hallmark of a spontaneously broken symmetry. The pion which is almost massless is identified with this Goldstone boson and its low mass reflects the non vanishing mass of the up and down quarks.

Even in the case of a vanishing bare quark mass, the vacuum is not invariant with respect to chiral symmetry. This breakdown of the chiral symmetry is dynamical and is reflected through the fact that quarks develop a constituent mass. This phenomenon occurs within the NJL model and is its raison-d'être.

As can be seen in the interaction Lagrangian, the couplings in the vector and axial channels are chosen to be equal to reproduce the near degenerate masses $\rho(770)$, $\omega(782)$ and $a_1(1260)$, $f_1(1285)$. Different couplings in these channels would anyway leave chiral symmetry unbroken.

$U_A(1)$ symmetry

The axial transformation $U_A(1)$ acts on the quark fields as follows

$$\psi \to \exp^{-i\gamma^5\theta}\psi\,, \qquad \bar{\psi} \to \bar{\psi}\exp^{-i\gamma^5\theta\tau}\,. \tag{6.5}$$

In the limit of vanishing quark masses (chiral limit $m \to 0$), the NJL Lagrangian is invariant under $U_A(1)$ transformation.

6.2.3 Masses and coupling constants in the NJL model

The interaction part of the NJL model describe quarks interacting via four-fermion interactions. The usual way to deal with this kind of interaction is to rewrite it in terms of boson fields that have the same transformation properties. After bosonisation, the quark fields are integrable and the theory contains only boson fields. From the Lagrangian thus obtained, it is straightforward to recognise the different masses and coupling constants. Here, we will follow an approach based on one- and two-body equations.

The gap equation

In order to determine the constituent quark mass, we solve the Schwinger-Dyson equation associated to the quark mass. For the NJL, model it can be pictorially represented by Fig. (6.1), where the thick line represents the full quark propagator and is parametrised by

$$S_F(p) = \frac{1}{\not{p} - m + i\epsilon}\,. \tag{6.6}$$

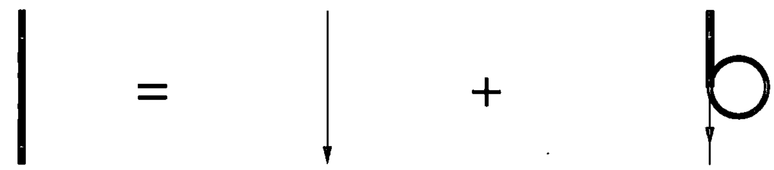

Figure 6.1: Schwinger-Dyson equation for the quark propagator

The Gap equation gives us an equation for the constituent mass, which reads

$$m = m_0 + mG_1N_c8i\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2-m^2}\,. \tag{6.7}$$

A cut-off Λ has been introduced to regularise the quadratic divergent loop integral. Thus m_0, G_1, G_2, Λ represent the parameters of the theory. If we compute the loop integral the Gap equation becomes

$$m = m_0 + \frac{G_1N_cm}{2\pi^2}\left(\Lambda^2 - m^2\log\left(1+\frac{\Lambda^2}{m^2}\right)\right)\,. \tag{6.8}$$

Even when $m_0 = 0$, if G_1 is sufficiently large the constituent mass m will be nonzero and chiral symmetry broken. In Fig. (6.2), we plot the constituent mass m for $\Lambda = 1.05$ GeV as a function of G_1 for $m_0 = 0, 0.005, 0.05$ GeV. For $m_0 = 0$ we see that G_1 has to be strong enough to generate a constituent mass m.

The Bethe-Salpeter equation

In order to investigate the meson fluctuations, we solve the Bethe-Salpeter equation associated to the $\mathcal{T}$ matrix. It reads Fig. (6.3)

$$\mathcal{T}(q) = \mathcal{K} + itr\int\frac{d^4p}{(2\pi)^4}\left[\mathcal{K}S_F\left(p+\frac{1}{2}q\right)\mathcal{T}(q)S_F\left(p-\frac{1}{2}q\right)\right]\,, \tag{6.9}$$

where $\mathcal{K}$ is the colour singlet two-body interaction Kernel, which can be decomposed into flavour and Lorentz tensor covariants in the quark-antiquark channel

$$\begin{aligned}\mathcal{K} \;=\; & K^S_{ij}(\lambda_i\otimes\lambda_j) + K^P_{ij}(i\gamma^5\lambda_i\otimes i\gamma^5\lambda_j)\\ & + K^V_{ij}(i\gamma_\mu\lambda_i\otimes i\gamma^\mu\lambda_j) + K^A_{ij}(\gamma_\mu\gamma^5\lambda_i\otimes\gamma^\mu\gamma^5\lambda_j)\end{aligned} \tag{6.10}$$

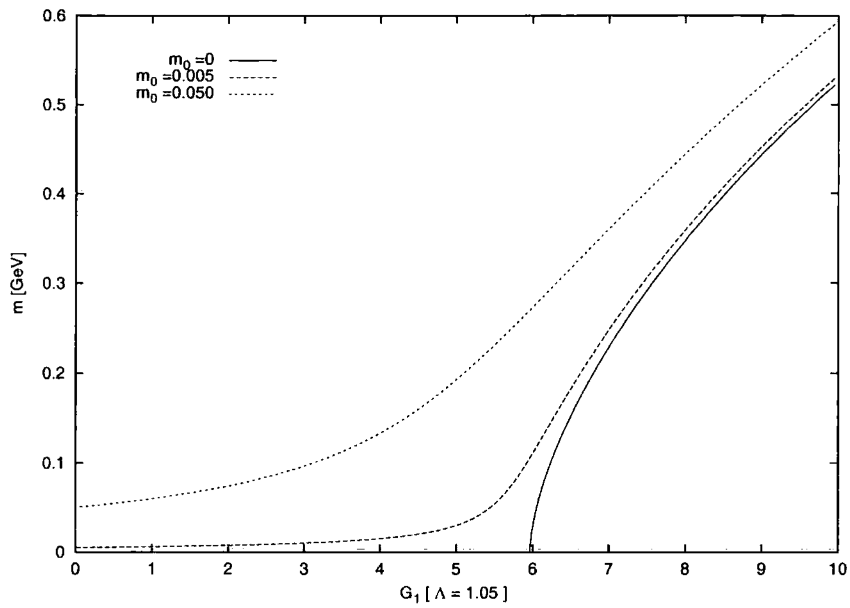

Figure 6.2: The current mass m as a function of the coupling G_1 for $m_0 = 0, 0.005, 0.05$ Gev

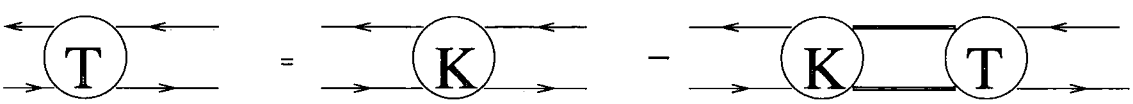

Figure 6.3: Schwinger-Dyson equation for the $\mathcal{T}$ matrix

The K_{ij}^{α} are given in terms of the couplings G_1 and G_2 and possibly others, had we included an anomaly term. In order to solve this equation, we decompose the $\mathcal{T}$ matrix in tensor invariants. We have

$$\mathcal{T}(q) = \mathcal{T}_{scalar}(q) + \mathcal{T}_{pseudoscalar}(q) + \mathcal{T}_{vector}(q) + \mathcal{T}_{axial}(q)\,, \tag{6.11}$$

where the scalar, pseudoscalar, vector and axial vector terms refer now to the corresponding mesonic modes, i.e. quark-antiquark states with spin and parity $J^{\pi} = 0^+, 0^-, 1^+, 1^-$, respectively. Each term can be decomposed into Lorentz covariant tensors as follows

$$\mathcal{T}_{scalar}(q) \;=\; M_{SS}(1 \otimes 1) + M_{SV}(1 \otimes i\hat{\not q})$$

$$
\begin{aligned}
& +M_{VS}(-i\hat{\not q}\otimes 1) + M_{VV}^{L}L^{\mu\nu}(\gamma_\mu\otimes\gamma_\nu)\,, & (6.12)\\
\mathcal{T}_{pseudoscalar}(q) &= M_{PP}(i\gamma^5\otimes i\gamma^5) + M_{PA}(i\gamma^5\otimes i\hat{\not q}\gamma^5)\\
& +M_{AP}(-i\hat{\not q}\gamma^5\otimes i\gamma^5) + M_{AA}^{L}(-i\hat{\not q}\gamma^5\otimes i\hat{\not q}\gamma^5)\,, & (6.13)\\
\mathcal{T}_{vector}(q) &= T^{\mu\nu}M_{VV}^{T}(\gamma_\mu\otimes\gamma_\nu)\,, & (6.14)\\
\mathcal{T}_{axial}(q) &= T^{\mu\nu}M_{AA}^{T}(\gamma_\mu\gamma^5\otimes\gamma_\nu\gamma^5)\,, & (6.15)
\end{aligned}
$$

where

$$L^{\mu\nu} = \hat{q}^\mu\hat{q}^\nu \qquad T^{\mu\nu} = g^{\mu\nu} - q^\mu q^\nu/q^2 \tag{6.16}$$

$$\hat{q}^\mu = q^\mu/\sqrt{q^2} \qquad \not q = \gamma_\mu q^\mu \tag{6.17}$$

In order to simplify the task of solving the Bethe-Salpeter equation for the $\mathcal{T}$ matrix, we decompose the kernel $\mathcal{K}$ on a new basis. We write

$$\mathcal{K}(q) = \mathcal{K}_{scalar}(q) + \mathcal{K}_{pseudoscalar}(q) + \mathcal{K}_{vector}(q) + \mathcal{K}_{axial}(q), \tag{6.18}$$

where

$$
\begin{aligned}
\mathcal{K}_{pseudoscalar}(q) &= K_{ij}^{P}(i\gamma^5\lambda_i\otimes i\gamma^5\lambda_j) + K_{ij}^{A}(-i\gamma_\mu\gamma^5\hat{q}^\mu\lambda_i\otimes i\gamma_\nu\gamma^5\hat{q}^\nu\lambda_j)\,,\\
\mathcal{K}_{scalar}(q) &= K_{ij}^{S}(1\lambda_i\otimes 1\lambda_j) + K_{ij}^{V}(-i\gamma_\mu\hat{q}^\mu\lambda_i\otimes i\gamma_\nu\hat{q}^\nu\lambda_j)\,,\\
\mathcal{K}_{vector}(q) &= T^{\mu\nu}K_{ij}^{V}(\gamma_\mu\lambda_i\otimes\gamma^\nu\lambda_j)\,,\\
\mathcal{K}_{axial}(q) &= T^{\mu\nu}K_{ij}^{A}(\gamma_\mu\gamma^5\lambda_i\otimes\gamma^\nu\gamma^5\lambda_j)\,. & (6.19)
\end{aligned}
$$

Using this basis and formally writing

$$\mathcal{T} = M(\Gamma,\Gamma')_{ij}(\Gamma\lambda_i\otimes\Gamma'\lambda_j)\,, \tag{6.20}$$

permits us to write a matrix equation for T, where M is a block-diagonal matrix and Eq. (6.9) can be written as four independent matrix equations in the pseudoscalar, scalar, vector, axial channels. The matrix equation in flavour and Dirac space obtained from Eq. (6.9) and using Eq. (6.20) is

$$M = K\,[1 + JM] = K\,[1 - JK]^{-1}\ , \tag{6.21}$$

where J refers to the following fermion loop

$$J(\Gamma, \Gamma')_{ij} = iN_c \, tr \int \frac{d^4p}{(2\pi)^4} \left[\Gamma \lambda_i S_F \left(p + \frac{1}{2}q\right) \Gamma' \lambda_j S_F \left(p - \frac{1}{2}q\right)\right] . \tag{6.22}$$

In our case, we have degenerate masses for the up and down quark and the calculation of J is thus simplified. It can be evaluated in terms of the integrals

$$\begin{aligned} I_1(m) &= i8N_c \int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} , && (6.23) \\ &= \frac{N_c}{2\pi^2} \left[\Lambda^2 - m^2 \ln\left(1 + \frac{\Lambda^2}{m^2}\right)\right] , && (6.24) \end{aligned}$$

and

$$\begin{aligned} I_2(q^2, m) &= i4N_c \int^{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{(p + \frac{1}{2}q)^2 - m^2 + i\epsilon} \frac{1}{(p - \frac{1}{2}q)^2 - m^2 + i\epsilon} , && (6.25) \\ &= \frac{N_c}{4pi^2} \int_0^1 dx \left[\frac{\Lambda^2}{\Lambda^2 + y} + \ln\left(\frac{y}{y + \Lambda^2}\right)\right] , && (6.26) \end{aligned}$$

$$I_2^0(m) = I_2(q^2 = 0, m) , \tag{6.27}$$

where

$$y(x, q^2) = q^2(x^2 - x) + m^2 . \tag{6.28}$$

In our case, we just want to consider the case of the π and ρ mesons, so the relevant J's are

$$\begin{aligned} J_{VV}^{(T)} T_{\nu}^{\mu} &= J(T_{\nu\beta}\gamma^{\beta}, T^{\mu\alpha}\gamma_{\alpha}) , \\ J_{VV}^{(T)} = J_{VV} &= \frac{2}{3}\left[(2m^2 + q^2)I_2 - 2m^2 I_2^0\right] , && (6.29) \\ J_{PP} = J(i\gamma^5, i\gamma^5) &= I_1(m) - q^2 I_2 , && (6.30) \\ J_{PA} = J_{AP} = J(i\gamma^5, i\gamma^5) &= 2m\sqrt{q^2} I_2 , && (6.31) \\ J_{AA} = J(i\gamma_{\nu}\gamma^5 \hat{q}^{\nu}, -i\gamma_{\mu}\gamma^5 \hat{q}^{\mu}) &= -4m^2 I_2 . && (6.32) \end{aligned}$$

Because of the explicit chiral symmetry breaking by the current quark mass m_0, there is a mixing between pseudoscalar and longitudinal axial fields, called $\pi - a_1$ mixing.

Therefore in the pseudoscalar channel, we really have a 2×2 matrix equation. The matrix $\mathcal{K}$ is

$$\mathcal{K}_\pi = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}, \tag{6.33}$$

and the matrix J is

$$J_\pi = \begin{pmatrix} J_{PP} & J_{PA} \\ J_{AP} & J_{AA} \end{pmatrix}. \tag{6.34}$$

The matrix M is

$$M_\pi = \begin{pmatrix} M_{PP} & M_{PA} \\ M_{AP} & M_{AA} \end{pmatrix}. \tag{6.35}$$

We obtain the M_π matrix for the pion from the equation Eq. (6.21). The information we want is M^π_{PP}, it is

$$M^\pi_{PP} = \frac{1}{D_\pi(q)} G_1 \left(1 - G_2 J_{AA}(q^2)\right), \tag{6.36}$$

$$\begin{aligned} D_\pi(q^2) &= \det\left(1 - J_\pi \mathcal{K}_\pi\right), \\ &= \left(1 - G_1 J_{PP}(q^2)\right)\left(1 - G_2 J_{AA}(q^2)\right) \\ &\quad - G_1 G_2 J_{AP}(q^2) J_{PA}(q^2). \end{aligned} \tag{6.37}$$

We also obtain the ratio

$$\frac{M^\pi_{PA}}{M^\pi_{PP}} = \frac{G_2}{1 - G_2 J_{AA}}. \tag{6.38}$$

Concerning the ρ meson, as we do not treat the six-fermion interaction term in our Lagrangian, the situation is much simpler. We have

$$K_\rho = G_2, \tag{6.39}$$

$$J_\rho = J_{VV}(q^2), \tag{6.40}$$

$$M_{VV} = \frac{G_2}{1 - G_2 J_{VV}(q^2)} = \frac{G_2}{D_\rho(q^2)}. \tag{6.41}$$

Now that we have determined the two $\mathcal{T}$ matrices, we can extract the masses, as they appear in the $\mathcal{T}$ matrix as poles. We can also determine the couplings of the π and ρ mesons to $\bar{q}q$, which correspond to the residue at the poles. Close to a pole we parametrise the $\mathcal{T}$ matrix as follows [31]

$$i\mathcal{T}_\pi(q^2) \approx \left(i\gamma^5\tau^a \otimes i\gamma^5\tau^a\right) ig_{\pi-\bar{q}q}(1 + a_\pi \hat{q}) \frac{i}{q^2 - m_\pi^2} ig_{\pi-\bar{q}q}(1 - a_\pi \hat{q}), \tag{6.42}$$

$$i\mathcal{T}_\rho(q^2) \approx (\gamma_\mu\tau^a \otimes \gamma_\nu\tau^a)\, ig_{\rho-\bar{q}q}\frac{i(g^{\mu\nu} - q^\mu q^\nu/q^2)}{q^2 - m_\rho^2} ig_{\rho-\bar{q}q}\,. \tag{6.43}$$

An expansion of the $\mathcal{T}$ matrix obtained in Eq. (6.38 - 6.41), around the pole permits us to determine the couplings $g_{\pi-\bar{q}q}$ and $g_{\rho-\bar{q}q}$. We have

$$g^2_{\pi-\bar{q}q} = -\left.\frac{G_1(1 - G_2 J_{AA})}{dD_\pi(q^2)/dq^2}\right|_{q^2=m_\pi^2}, \tag{6.44}$$

$$a_\pi = \frac{1}{m_\pi}\frac{M^\pi_{PA}}{M^\pi_{PP}} = \left.\frac{G_2}{1 - G_2 J_{AA}}\right|_{q^2=m_\pi^2}, \tag{6.45}$$

$$g^2_{\rho-\bar{q}q} = \left.\frac{G_2}{dD_\rho(q^2)/dq^2}\right|_{q^2=m_\rho^2}. \tag{6.46}$$

We finally reached the point we wished. Starting from the NJL model, with four parameters, we can determine the π and ρ properties.

6.2.4 Meson masses as a function of quark mass

Polleri & al. [31], set their goal to show that the NJL model is able to determine the properties of the ρ meson. They use the following parameters

$$\begin{array}{ll} \Lambda = 1.05 \quad \text{GeV}, & G_1\Lambda^2 = 10.1, \\ m_0 = 3.33 \quad \text{MeV}, & G_2\Lambda^2 = -14.4, \end{array} \tag{6.47}$$

with which they are able to fit the quark condensate $< \bar{q}q >$, the pion mass m_π, the ρ meson mass m_ρ and the pion decay constant f_π. The values determined in this way are

$$\begin{array}{ll} < \bar{q}q >^{1/3} = -293 \quad \text{MeV}\,, & m_\pi = 139 \quad \text{MeV}\,, \\ m_\rho^{(0)} = 834 \quad \text{MeV}\,, & f_\pi = 93 \quad \text{MeV}\,. \end{array} \tag{6.48}$$

The prediction for the other quantities is

$$m = 463 \quad \text{MeV}\,, \tag{6.49}$$

$$g_{\pi-\bar{q}q} = 4.94\,, \tag{6.50}$$

$$a_\pi = 0.46\,, \tag{6.51}$$

$$g_{\rho-\bar{q}q} = 2.12\,. \tag{6.52}$$

Using these parameters, they proceed to compute the $\rho \to \pi\pi$ decay amplitude and its contribution to the self energy of the ρ. They find

$$\Gamma_{\rho\to\pi\pi} = 118 \quad \text{MeV}\,, \tag{6.53}$$

and a mass shift of -64 MeV , which produces

$$m_\rho^2 = (m_\rho^{(0)})^2 + \Re e\, \Sigma^\rho_{\pi\pi}\left((m_\rho^{(0)})^2\right) = 770^2 \quad \text{MeV}^2\,. \tag{6.54}$$

We now embark on a study of the meson masses as a function of the current quark mass. Using the parameters in Eq. (6.47), we plot the tree level meson masses in Fig. (6.4) ($\mathbf{m_q}$ is m_0).

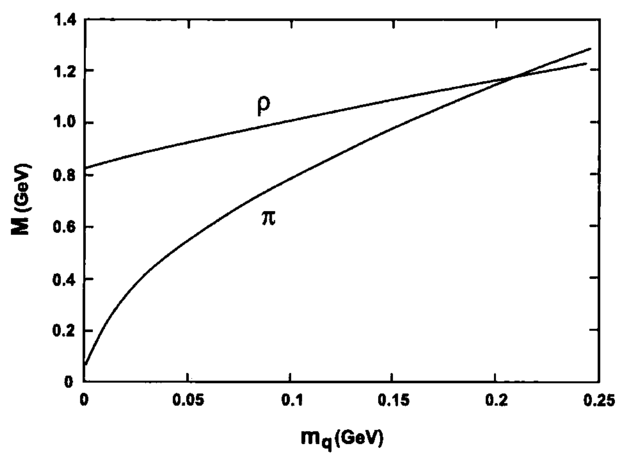

Figure 6.4: π and ρ (dash) meson mass in GeV as a function of current quark mass in Gev

6.2.5 Meson masses on the lattice

Our goal is to compare the meson masses obtained from the NJL calculation and the one computed on the lattice. Our data are obtained from the CP-PACS group [32]. The simulation is a full QCD computation using improved action. Hadron masses are obtained from the propagators computed on a $16^3 \times 32$ lattice with $\beta = 1.9$. The lattice spacing a, is determined through the determination of the string tension σa^2, that is fitted according to

$$\sigma a^2 = A_\sigma + B_\sigma (m_{PS}a)^2 + C_\sigma (m_{PS}a)^3, \tag{6.55}$$

using $\sqrt{\sigma} = 440$ MeV in the chiral limit. In Fig. (6.5), we plot the pion and ρ mass obtained with the NJL model with the parameter of Eq. (6.47) and the lattice data. Already with these parameters, the pion mass agrees well with the lattice data. The

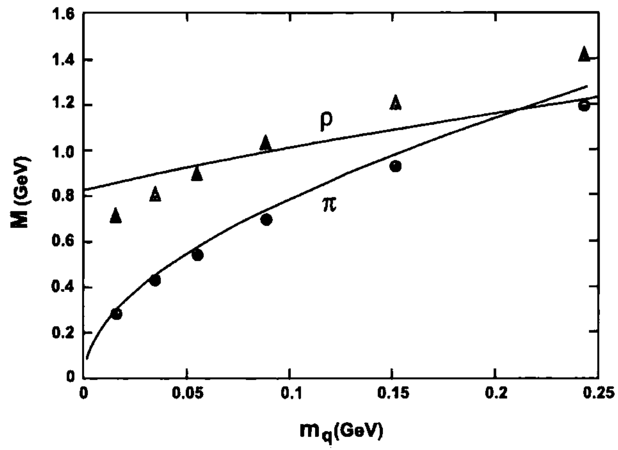

Figure 6.5: pion and ρ meson lattice data compared with NJL calculation

ρ mass fails to agree, does not have the right slope and even becomes lighter than the pion at high quark mass.

6.2.6 Fits

In this section we proceed to fit the lattice data with the NJL calculation. We have three parameters to use, namely Λ, G_1, and G_2. With the parameters

$$\Lambda = 0.9972 \quad \text{GeV}, \tag{6.56}$$

$$G_1 = 10.5/\Lambda^2, \tag{6.57}$$

$$G_2 = -12/\Lambda^2, \tag{6.58}$$

we achieve a good fit for the pion and use the same parameters to compute the ρ meson mass. As can one see on Fig. (6.6), the quark mass dependence of the ρ meson mass does not agree with the lattice data. The NJL model fails to reproduce the right slope. The mass of the pion $m_\pi = 139$ MeV is reproduced for $m_0 = 3.786$ MeV. For this particular quark current mass the calculated ρ mass is

$$m_\rho^{(0)} = 867 \quad \text{MeV}, \tag{6.59}$$

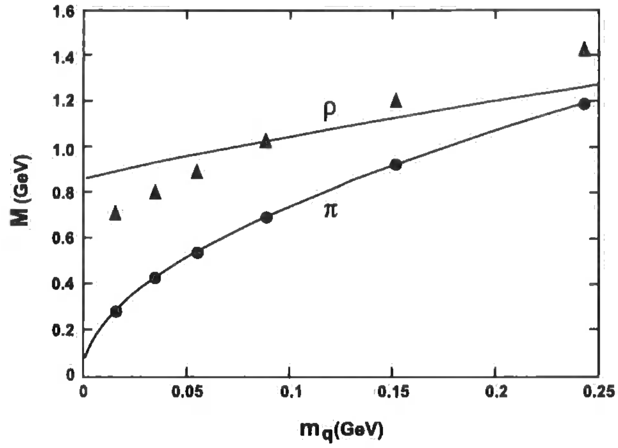

Figure 6.6: fitted pion mass and predicted ρ mass (dash) in the NJL model

and the dressed ρ has mass

$$m_\rho = 803 \quad \text{MeV}, \tag{6.60}$$

where we have used the mass shift $\Delta M = -64$ MeV as quoted before. This predicted ρ mass does not agree with the experimental value $m_\rho = 770$ MeV. The value of f_π for this current mass is found to be $f_\pi = 93.06$ MeV. A fit of the ρ meson mass alone was attempted but failed. It thus seems that the finding of the right ρ meson mass from the NJL model is fortuitous.

We now investigate whether it is possible to achieve a better fit for the ρ meson by introducing a fourth parameter. In the Gap and the Bethe-Salpeter equations, we have included a cut-off Λ, which is of the order 1 GeV. For high quark mass, the meson masses reach and even go above this value. In order to take into account the propagation of higher masses in the loop of the BS equation, we thus may need to increase the value of the cut-off. We thus try to fit the lattice data using the four following parameters: $\Lambda_0, \alpha_0, g_1, g_2$, which define the parameters of the theory

$$\begin{aligned} \Lambda &= \Lambda_0 + \alpha_0 m_0 \,, \\ G_1 &= 10\frac{g_1}{\Lambda^2} \,, \\ G_2 &= -10\frac{g_2}{\Lambda^2} \,. \end{aligned} \tag{6.61}$$

We use the data obtained by the CP-PACS group [33], which is a better simulation than the one used previously. As the authors mention in [33], the study [32] was a preparatory work concerning the simulation of full QCD using improved actions. As can be seen in Fig. (6.5), the quark mass ranges from 15 to 250 MeV, whereas in Fig. (6.7), the range is smaller ([50, 150]), but we can consider this simulation to be more reliable since it has more data points, which correspond to high quark masses. In Fig. (6.5), we can check that the first point, corresponding to $m_0 = 15$ MeV is unphysical since it predicts a ρ mass that is smaller than the physical ρ. A good fit for the pion is obtained and gives a null value for the parameter α_0. The prediction for the ρ meson mass calculated using the parameters found for the π, and a total disagreement is found again as can be seen in Fig. (6.7).

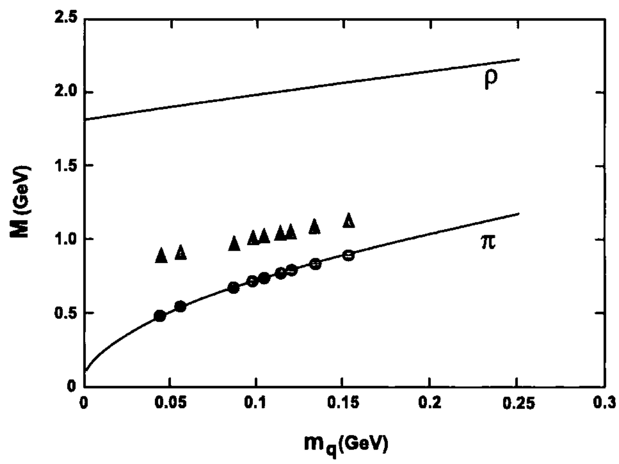

Figure 6.7: fitted pion mass and predicted ρ mass in the NJL model to compare with ρ mass obtained from the lattice

The four parameter fit for the ρ is achievable and gives us

$$\Lambda_0 = 0.9975\,, \tag{6.62}$$

$$\alpha_0 = 0.9294\,, \tag{6.63}$$

$$g_1 = 0.9946\,, \tag{6.64}$$

$$g_2 = 1.6201\,. \tag{6.65}$$

$$\tag{6.66}$$

In Fig. (6.8), we show the fit for the ρ, as well as the prediction for the π using these

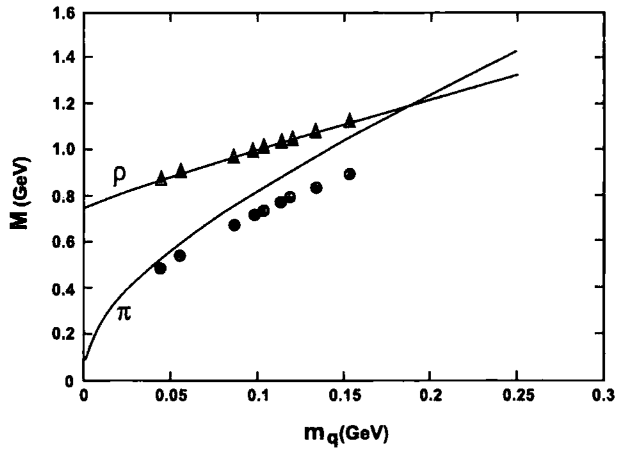

Figure 6.8: fitted ρ mass and predicted π mass in the NJL model

parameters. The good fit for the pion is now lost and the inconsistency of the quark mass dependence between the ρ and the π is not resolved.

6.3 A New Functional Approach

In this section, we will introduce a functional approach to tackle the same problem of bound state masses. In principle this technique has a much wider scope and indeed has been used in very different contexts. We will apply this method to the problem of bound state masses to show how one could study dynamical mass generation from a new perspective. This approach enables us to write an exact functional integral equation for the effective action Γ which is also the generating functional of the 1PI n-point functions. It is also related to the Callan-Symanzik equations (CS) which in perturbation theory is an equation giving the dependence of the n-point functions with respect to the bare mass. What we propose to do is to generalise this CS method and derive an exact equation for the effective action directly from its definition [35, 36, 37, 38]. As this is our first endeavour to apply this technique, we will only consider the NJL model with the scalar interaction.

6.3.1 Evolution equation

We consider the Lagrangian of the NJL model given by

$$\mathcal{L} = \bar{\Psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - zm_0\right)\Psi(x) + \frac{g}{2}\left(\bar{\Psi}(x)\Psi(x)\right)^2 . \tag{6.67}$$

The parameter z is introduced to control the amplitude of the fluctuations. For $z >> 1$, the Lagrangian describes a heavy fermion and the theory is perturbative since the mass term dominates the action. As z decreases the interaction term becomes more and more important and quantum corrections increase in magnitude. Our purpose is to study the dependence on z of the effective action Γ_z, the generating functional of proper graphs.

The Lagrangian, we have written contains a four-fermion interaction term, which is not suitable for analytical treatment. We rewrite the Lagrangian by introducing a field Φ, which has the same transformation properties as the bilinear term $\bar{\Psi}(x)\Psi(x)$. The Lagrangian becomes

$$\mathcal{L} = \bar{\Psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - zm_0 + \phi\right)\Psi(x) - \frac{1}{2g}\Phi^2(x) . \tag{6.68}$$

The functional $W_z\left[\bar{\eta}, \eta, j\right]$ which generates the connected graphs is given by

$$\exp W_z\left[\bar{\eta}, \eta, j\right] = \int \mathcal{D}\left[\bar{\Psi}, \Psi, \Phi\right] \exp\left\{ i\int_x \mathcal{L} + i\int_x \left(j\Phi + \bar{\eta}\Psi + \bar{\Psi}\eta\right)\right\} . \tag{6.69}$$

The derivatives of W_z with respect to the sources $\bar{\eta}, \eta, j$ are given by

$$\begin{aligned} \frac{\delta}{\delta\bar{\eta}} W_z &= i < \Psi >= i\psi , & (6.70)\\ W_z \frac{\overleftarrow{\delta}}{\delta\eta} &= i < \bar{\Psi} >= i\bar{\psi} , & (6.71)\\ \frac{\delta}{\delta j} W_z &= i < \Phi >= i\phi . & (6.72) \end{aligned}$$

We also have

$$\left(\frac{\delta}{\delta\bar{\eta}} W_z \frac{\overleftarrow{\delta}}{\delta\eta}\right) = -\bar{\psi}\psi + < \bar{\psi}\psi > . \tag{6.73}$$

The effective action $\Gamma_z\left[\bar{\psi}, \psi, \phi\right]$ is the Legendre transform of W_z

$$W_z = i\Gamma_z + i\int_x \left(j\phi + \bar{\eta}\psi + \bar{\psi}\eta\right) , \tag{6.74}$$

with functional derivatives

$$\frac{\delta}{\delta\bar{\psi}}\Gamma_z = -\eta\,, \tag{6.75}$$

$$\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi} = -\bar{\eta}\,, \tag{6.76}$$

$$\frac{\delta}{\delta\phi}\Gamma_z = -j\,. \tag{6.77}$$

We also have a relation between the second derivative of W_z and the second derivative of Γ_z, which is

$$-i\left(\frac{\delta}{\delta\bar{\psi}}\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi}\right)^{-1} = \left(\frac{\delta}{\delta\bar{\eta}}W_z\frac{\overleftarrow{\delta}}{\delta\eta}\right)\,. \tag{6.78}$$

To compute the derivative of Γ_z with respect to z, one has to remember that the independent variables are the the fields $\bar{\psi}$, ψ and ϕ and the parameter z. We therefore have

$$\begin{aligned}
\partial_z\Gamma_z &= -i\partial_z W_z - i\int\left(\partial_z\bar{\eta}\frac{\delta W_z}{\delta\bar{\eta}} + W_z\frac{\overleftarrow{\delta}}{\delta\eta}\partial_z\bar{\eta} + \frac{\delta W_z}{\delta j}\partial_z j\right) \\
&\quad - \int\left(\bar{\psi}\partial_z\eta + \partial_x\bar{\eta}\psi + \partial_z j\right)\,, \\
&= -i\partial_z W_z\,,
\end{aligned} \tag{6.79}$$

after using the functional derivative of W_z in Eq. (6.70-6.72). The derivative of W_z with respect to z is

$$\begin{aligned}
-i\partial_z W_z &= -m_0 < \int_x \bar{\psi}\psi >\,, \\
&= -m_0\int_x \bar{\psi}\psi - m_0\int_x\left(\frac{\delta}{\delta\bar{\eta}}W_z\frac{\overleftarrow{\delta}}{\delta\eta}\right)\,,
\end{aligned} \tag{6.80}$$

which after using Eq. (6.78)can be rewritten as

$$\partial_z\Gamma_z\left[\bar{\psi},\psi,\phi\right] + m_0\int_x\bar{\psi}\psi = i\,m_0\,\mathrm{Tr}\left(\frac{\delta}{\delta\bar{\psi}}\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi}\right)^{-1}\,, \tag{6.81}$$

where Tr denotes the trace over spacetime and Dirac indices. The inverse matrix $[\delta^2\Gamma]^{-1}$ has to be taken with respect to field variables, spacetime indices and Dirac indices. We note that Eq. (6.81) is an exact equation since we have not yet adopted any approximation. We can also note that Eq. (6.81) has the same form as the one derived in exact Renormalisation Group (RG) methods [39] . A similar equation has already been derived for the case of QED in $d = 4 - \epsilon$ dimensions, which reproduced the usual one loop behaviour for the beta function but, has the advantage of avoiding the appearance of the Landau pole (in a specific ansatz for the effective action) by taking into account the running of the mass. This avoidance of the Landau pole by a running mass is the same way the Landau pole is claimed to be avoided in lattice QED [40]. Also the case of QED in an external field was treated using this technique and it was shown non-perturbatively how the full fermion propagator and the full vertex depend on the external gauge field.

Before proceeding to the treatment of the functional integral equation Eq. (6.81), we first show how it can be rewritten as a first order functional integral equation for the effective action Γ_z. As was mentioned when deriving Schwinger-Dyson equations in section. (2.2), the functional integral of a functional derivative is zero. We can write

$$0 = \int \mathcal{D}\left[\bar{\Psi}, \Psi, \Phi\right] \frac{\delta}{\delta\phi} \exp\left\{i \int_x \mathcal{L} + i \int_x \left(j\Phi + \bar{\eta}\Psi + \bar{\Psi}\eta\right)\right\}, \tag{6.82}$$

which gives us

$$< \bar{\Psi}\Psi > - \frac{1}{g} < \Phi > + j = 0\,, \tag{6.83}$$

or after using the first derivative of the effective action Γ_z in Eq. (6.77)

$$\begin{aligned} \frac{\delta\Gamma_z}{\delta\phi} &= < \bar{\Psi}\Psi > - \frac{1}{g}\phi\,, \\ &= -i \left(\frac{\delta}{\delta\bar{\psi}} \Gamma_z \frac{\overleftarrow{\delta}}{\delta\psi}\right)^{-1} + \bar{\psi}\psi - \frac{1}{g}\phi\,. \end{aligned} \tag{6.84}$$

This last relation permits us to rewrite the second order functional integral equation Eq. (6.81) as

$$\partial_z \Gamma_z \left[\bar{\psi}, \psi, \phi\right] + \frac{m_0}{g} \int_x \phi = -m_0 \int_x \frac{\delta\Gamma_z}{\delta\phi}\,. \tag{6.85}$$

If we differentiate the last equation Eq. (6.85) with respect to z, we obtain

$$\partial_z^2 \Gamma_z = (-m_0)^2 \int_{x_1, x_2} \frac{\delta^2\Gamma_z}{\delta\phi(x_1)\delta\phi(x_2)}\,, \tag{6.86}$$

and the n^{th} derivative is

$$\partial_z^n \Gamma_z = (-m_0)^2 \int_{x_1,\ldots,x_n} \frac{\delta^n}{\delta\phi(x_1)\ldots\delta\phi(x_n)} \Gamma_z \,. \tag{6.87}$$

We now are able to make the resummation

$$\Gamma_z\left[\bar{\psi},\psi,\phi\right] = \sum_{n=0}^{n=\infty} \frac{z^n}{n!}\frac{\delta\Gamma_z}{\delta z}\bigg|_{z=0} , \tag{6.88}$$

which is, after using Eq. (6.87)

$$\Gamma_z\left[\bar{\psi},\psi,\phi\right] = \sum_{n=0}^{n=\infty} \frac{(-zm_0)^n}{n!} \frac{\delta^n}{\delta\phi(x_1)\ldots\delta\phi(x_n)} \Gamma_z|_{z=0} - \frac{m_0}{g}\int\phi \tag{6.89}$$

$$= \exp^{\left(-zm_0 \frac{\delta}{\delta\phi}\right)} \Gamma_z\left[\bar{\psi},\psi,\phi\right]\Big|_{z=0} - \frac{m_0}{g}\int\phi \,. \tag{6.90}$$

In the last equation $\Gamma_z\left[\bar{\psi},\psi,\phi\right]\Big|_{z=0}$ is the effective action in the chiral limit since z is zero. We denote the chiral effective action as follows

$$\Gamma_z\left[\bar{\psi},\psi,\phi\right]\Big|_{z=0} = \Gamma_0\left[\bar{\psi},\psi,\phi\right] , \tag{6.91}$$

and write the equation for the effective action Γ_z as

$$\Gamma_z\left[\bar{\psi},\psi,\phi\right] = \Gamma_0\left[\bar{\psi},\psi,\phi - zm_0\right] - \frac{m_0}{g}\int\phi \,, \tag{6.92}$$

since we recognise in Eq. (6.90) the functional generalisation of the well known result

$$exp\left(a\frac{d}{dx}\right) f(x) = f(x+a) \,. \tag{6.93}$$

The effective action Γ_z for any z is therefore given by the chiral limit effective action Γ_0, where the scalar field ϕ is translated by the amount $-zm_0$. This mechanism is reminiscent of what is known in the background field methods [41]. This was also found for the case of *QED* in an external field, where it was shown in [36], using this functional technique, that the effective action in the presence of an external field A_μ^{ext} is the same as the one without the external field, but where the dynamical gauge field is translated by the vector A_μ^{ext}.

The interest of the first order functional integral equation is that it allows us to derive relations similar to Ward identities in gauge field theory. If we differentiate Eq. (6.85) with respect to $\bar{\psi}(x)$, then with respect to $\psi(y)$ we obtain

$$\frac{\delta}{\delta\bar{\psi}(x)}\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi(y)} = -m_0\int_z\frac{\delta}{\delta\phi(z)}\frac{\delta}{\delta\bar{\psi}(x)}\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi(y)} \tag{6.94}$$

We now define the full inverse propagator $S_z^{-1}(x,y)$ and the full $\bar{\psi}\phi\psi$ vertex $\Lambda_z(t;x,y)$ as

$$S_z^{-1}(x,y) = -i\left.\frac{\delta}{\delta\bar{\psi}(x)}\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi(y)}\right|_{\bar{\psi}=\psi=\phi=0}, \tag{6.95}$$

$$\Lambda_z(t;x,y) = \left.\frac{\delta}{\delta\phi(t)}\frac{\delta}{\delta\bar{\psi}(x)}\Gamma_z\frac{\overleftarrow{\delta}}{\delta\psi(y)}\right|_{\bar{\psi}=\psi=\phi=0}, \tag{6.96}$$

and are able to write a relation between the full inverse propagator S_z^{-1} and the full vertex Λ_z

$$-i\partial_z S_z^{-1}(x,y) = -m_0\int_{x_1}\Lambda_z(x_1;x,y)\,. \tag{6.97}$$

This last relation Eq. (6.97) can be seen as a kind of Ward identity except that it is integral rather than derivative. It involves the z derivative of the full inverse propagator S_z^{-1}. If we note that

$$\partial_z S_z^{-1}(x,y) = -\int_{x_1,x_2} S_z^{-1}(x,x_1)\,\partial_z S_z(x_1,x_2)\,S_z^{-1}(x_2,y)\,, \tag{6.98}$$

we can write a relation between the full propagator $S_z(x,y)$ and the full vertex $\Lambda_z(t;x,y)$

$$\partial_z S_z(x,y) + im_0\int_{x_1,x_2,x_3} S_z(x,x_2)\,\Lambda_z(x_1;x_2,x_3)\,S_z(x_3,y) = 0\,. \tag{6.99}$$

6.3.2 Gradient expansion

Up to know, we have been very formal and it is now desirable to show how this technique can be used to make predictions. Our main result is the functional integral equation Eq. (6.81). As it is written, it cannot be solved analytically and we therefore need an ansatz for the effective action Γ_z. We will make a gradient expansion to find an approximate solution to the evolution equation Eq. (6.81)

$$\Gamma_z\left[\bar{\psi},\psi,\phi\right] = \int_x \phi(x)\mathcal{D}^{-1}\phi(x) + \bar{\psi}(x)\mathcal{G}^{-1}\psi(x) + \lambda_z(x)\bar{\psi}(x)\phi(x)\psi(x)\,, \tag{6.100}$$

where $\mathcal{D}$ and $\mathcal{G}$ are the scalar and fermion propagators, respectively and $\lambda_z(x)$ is a coupling function. In the most general case the propagators in momentum space are

$$\begin{aligned}
\mathcal{D}^{-1}(p) &= \beta(z,p^2)p^2 - m_\phi(z,p^2)\,, & (6.101)\\
\mathcal{G}^{-1}(p) &= Z(z,p^2)\not{p} - m(z,p^2) - zm_0\,, & (6.102)
\end{aligned}$$

and as a first approximation, we can choose

$$\lambda_z(x) = \lambda(z)\,. \tag{6.103}$$

The evolution of the different functions β, Z and λ are obtained when we expand both sides of Eq. (6.81) in powers of the scalar field and fermion field and by identifying the operators on both sides. As we have already mentioned earlier, the calculation of the trace requires us to compute the inverse matrix $\left[\Gamma^{(2)}\right]^{-1}$, which is the inverse of the matrix of second derivatives

$$\Gamma^{(2)} = \begin{pmatrix} \frac{\delta\Gamma\overleftarrow{\delta}}{\delta\bar{\psi}\delta\psi} & \frac{\delta^2\Gamma}{\delta\bar{\psi}\delta\bar{\psi}} & \frac{\delta^2\Gamma}{\delta\phi\delta\bar{\psi}} \\ \frac{\Gamma\overleftarrow{\delta}^2}{\delta\psi\delta\psi} & \frac{\delta\Gamma\overleftarrow{\delta}}{\delta\bar{\psi}\delta\psi} & \frac{\delta\Gamma\overleftarrow{\delta}}{\delta\phi\delta\psi} \\ \frac{\delta\Gamma\overleftarrow{\delta}}{\delta\phi\delta\psi} & \frac{\delta^2\Gamma}{\delta\bar{\psi}\delta\phi} & \frac{\delta^2\Gamma}{\delta\phi\delta\phi} \end{pmatrix}. \tag{6.104}$$

We decompose $\Gamma^{(2)}$ into a diagonal $\Gamma^{(2)}_\Delta$ plus non-diagonal part $\Gamma^{(2)}_{nd}$ and compute $\left[\Gamma^{(2)}\right]^{-1}$ by expanding in the diagonal part $\Gamma^{(2)}_\Delta$ in momentum space

$$\begin{aligned}
\left[\Gamma^{(2)}\right]^{-1} = {} & \left[\Gamma^{(2)}_\Delta\right]^{-1} - \left[\Gamma^{(2)}_\Delta\right]^{-1}\Gamma^{(2)}_{nd}\left[\Gamma^{(2)}_\Delta\right]^{-1} \\
& + \left[\Gamma^{(2)}_\Delta\right]^{-1}\Gamma^{(2)}_{nd}\left[\Gamma^{(2)}_\Delta\right]^{-1}\Gamma^{(2)}_{nd}\left[\Gamma^{(2)}_\Delta\right]^{-1} \\
& + \left[\Gamma^{(2)}_\Delta\right]^{-1}\Gamma^{(2)}_{nd}\left[\Gamma^{(2)}_\Delta\right]^{-1}\Gamma^{(2)}_{nd}\left[\Gamma^{(2)}_\Delta\right]^{-1}\Gamma^{(2)}_{nd}\left[\Gamma^{(2)}_\Delta\right]^{-1} + \ldots\,, \qquad (6.105)
\end{aligned}$$

where

$$\Gamma^{(2)}_{nd} = \Gamma^{(2)} - \Gamma^{(2)}_\Delta\,. \tag{6.106}$$

In this way we obtain [37]

$$\left(\frac{\delta\Gamma \overleftarrow{\delta}}{\delta\bar{\psi}\delta\psi}\right)^{-1} = \mathcal{G}+\mathcal{G}\left(-c+c\mathcal{G}c-c\mathcal{G}c\mathcal{G}c+a\mathcal{D}\tilde{b}-a\mathcal{D}\tilde{b}\mathcal{G}c-c\mathcal{G}a\mathcal{D}\tilde{b}\right)\mathcal{G}+\ldots\,, \quad (6.107)$$

with [37]

$$a(p,q) = \lambda(z)\psi(-p-q)\,, \quad (6.108)$$

$$b(p,q) = \bar{\psi}(-p-q)\lambda(z)\,, \quad (6.109)$$

$$c(p,q) = -\lambda(z)\phi(-p-q)\,, \quad (6.110)$$

and the tilde denotes transposition in momentum and spinor spaces.

Following [37] the evolution equations are

$$\partial_z\mathcal{D}^{-1}(p) = -2im_0\lambda^2(z)\int_q \mathrm{Tr}\left[\mathcal{G}^2(q)\mathcal{G}(p+q)\right]\,, \quad (6.111)$$

$$\partial_z\mathcal{G}^{-1}(p)+m_0 = -im_0\lambda^2(z)\int_q \mathcal{D}(q-p)\mathcal{G}^2(q)\,, \quad (6.112)$$

$$\partial_z\lambda(z) = -2m_0\lambda^3(z)\int_q \mathcal{D}(q)\mathcal{G}^2(q)\mathcal{G}(q)\,. \quad (6.113)$$

A close look at these equations shows us that the integrals on the right hand sides are the usual terms that would appear in the Schwinger-Dyson equations except that the fermion propagator $\mathcal{G}(q)$ is squared. This can be understood easily since a change $zm_0 \to (z+\delta z)m_0$ of the bare fermion propagator changes the propagator as $\mathcal{G} \to \mathcal{G}+\mathcal{G}\delta z m_0\mathcal{G}$ in the internal lines of Feynman graphs. These equations are integro-differential equations and need further study to give results concerning the infra-red behaviour of the fermion and scalar propagators. This approach has already been adopted in [37], where the case of QED was considered and as already mentioned, with a simple ansatz for the photon and electron propagators, it was shown analytically that the Landau pole is avoided by incorporating a running mass in the evolution of the coupling.

6.3.3 Scalar theory

After bosonisation of the purely fermionic Lagrangian of Eq. (6.67), we have a theory involving a fermion field ψ interacting with a scalar field ϕ. The Lagrangian is quadratic in the fermion field and therefore the integration over the fermionic field is analytical. After integrating the fermionic fields, we obtain a purely scalar theory whose Euclidean Lagrangian is

$$\mathcal{L}(\phi) = -i\mathrm{Tr}\left[1 + (-i\,\not{\partial} + m)^{-1}\,\phi\right] + \frac{1}{2g}\phi^2\,, \tag{6.114}$$

where we have taken $z = 1$. We have seen previously that it was possible to write an evolution equation in z for the effective action because the operator that multiplies z is quadratic. As we would like to have a new way to study dynamical symmetry breaking, we would like to write an evolution equation in the coupling g. The parameter g only appears after bosonisation in front of the quadratic operator ϕ^2 and therefore we can apply the previous method and write an evolution equation in g for the effective action. It reads

$$\begin{aligned}
\partial_g \Gamma_g\,[\phi] &= -\partial_g W_g\,[j] \\
&= \frac{1}{2g^2}\int_x <\phi^2> \\
&= \frac{1}{2g^2}\int_x \left[\phi^2 - \frac{\delta^2 W_g}{\delta j \delta j}\right] \\
&= \frac{1}{2g^2}\int_x \phi^2 - \frac{1}{2g^2}\,\mathrm{Tr}\left[\frac{\delta^2\Gamma_g}{\delta\phi\delta\phi}\right]^{-1}.
\end{aligned} \tag{6.115}$$

The gradient expansion for the scalar theory will be

$$\Gamma_g\,[\phi] = \int_x Z(\phi)\partial_\mu\phi\partial^\mu\phi + V_g(\phi) + \dots\,, \tag{6.116}$$

where $Z(\phi)$ is the renormalisation function for the scalar field ϕ and $V_g(\phi)$ the effective potential. The term $\dots$ represents higher derivative contributions which we neglect. We work in the Local Potential Approximation $Z(\phi) = 1$ and consider a constant configuration for the field $\phi = \phi_0$, which is enough to obtain an evolution equation for the effective potential $V_g(\phi)$. We have

$$\Gamma_g\,[\phi_0] = \mathcal{V} V_g(\phi_0)\,, \tag{6.117}$$

$$\left(\frac{\delta^2\Gamma_g}{\delta\phi(p)\delta\phi(q)}\right)^{-1} = \frac{1}{p^2 + V_g''(\phi)}\delta^3(p+q)\,, \tag{6.118}$$

where $\mathcal{V}$ is the four-dimensional volume and $'$ denotes a differentiation with respect to the argument. The evolution equation for the effective potential is thus

$$\partial_g V_g(\phi) - \frac{1}{2g^2}\phi^2 = -\frac{1}{2g^2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 + V_g''(\phi)} \,. \tag{6.119}$$

We introduce a cut-off Λ^2 in to regulate the integral and obtain

$$\partial_g V_g(\phi) - \frac{1}{2g^2}\phi^2 = -\frac{1}{32g^2\pi^2}\left[\Lambda^2 - V_g''(\phi)\ln\left(1 + \frac{\Lambda^2}{V_g''(\phi)}\right)\right] \,. \tag{6.120}$$

This partial differential equation contains the evolution equation for all the different couplings since the effective potential $V_g(\phi)$ can be Taylor expanded in ϕ

$$V_g(\phi) = \sum_{n=0}^{n=\infty} c_n \phi^n \,, \tag{6.121}$$

where the c_n are coupling constants. We assume here that the potential has only two relevant couplings $m_g = m$ and $\lambda_g = \lambda$ and write

$$V_g(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \,, \tag{6.122}$$

$$V_g''(\phi) = m^2 + \frac{1}{2}\lambda\phi^2 \,. \tag{6.123}$$

In order to obtain the evolution equations for the couplings m and λ, we expand the integrand in Eq. (6.119) in ϕ

$$\frac{1}{p^2 + m^2 + \frac{\lambda}{2}\phi^2} = \frac{1}{p^2 + m^2} - \frac{\lambda}{2}\frac{1}{(p^2+m^2)^2}\phi^2 + \frac{\lambda^2}{4}\frac{1}{(p^2+m^2)^3}\phi^4 \,. \tag{6.124}$$

By identifying the operators from the two sides of Eq. (6.119), we obtain the following evolution equations for m and λ

$$\frac{1}{2}\partial_g\left(m^2\right) - \frac{1}{2g^2} = \frac{\lambda}{4g^2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^2} \,, \tag{6.125}$$

$$\frac{1}{4!}\partial_g\lambda = -\frac{\lambda^2}{8g^2}\int \frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m^2)^2} \,, \tag{6.126}$$

which becomes after integration

$$\frac{1}{2}\partial_g\left(m^2\right) - \frac{1}{2g^2} = \frac{\lambda}{64\pi^2 g^2}\left[\ln\left(\frac{m^2+\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{m^2+\Lambda^2}\right] \,, \tag{6.127}$$

$$\partial_g\lambda = -\frac{3\lambda^2}{32\pi^2 g^2}\frac{\Lambda^4}{m^2\left(m^2+\Lambda^2\right)^2} \,. \tag{6.128}$$

The equation for m needs regulation but in theory the equation for λ is free of divergence since its associated integral is convergent. For $\Lambda = \infty$ Eq. (6.128) becomes

$$\partial_g \lambda = -\frac{3}{32\pi^2}\frac{\lambda^2}{g^2 m^2}\,, \tag{6.129}$$

but as we have an effective theory we need to treat Λ^2 as a parameter of the theory as we did in the standard NJL model.

6.4 Conclusion

We have presented in this chapter a well-known method for the treatment of bound states (NJL model) as well as a new approach [35, 36, 37, 38], which could be called functional Callan-Symanzik approach if the fluctuations are controlled by the mass parameter or by some other name when the control parameter in the Lagrangian is not the mass. In order to obtain a tractable equation, we made a gradient expansion of the effective action Γ, which is more localised in space than the Lagrangian. In the NJL model, it is the Lagrangian that is expanded and the expansion produces the different masses and couplings. However, this theoretical approach is limited since in practice, when one tries to compute the mass of a meson, say the pion, we have to adjust the parameters so as to be able to find a solution. Indeed, the polarisation function $I_2(q^2)$, which appears in the calculation diverges for $q^2 > 4M^2$, where M is the current quark mass. For the pion, all is well but when one treats the $a_1(1235)$, the equation

$$m_{a_1}(q^2) = q^2\,, \tag{6.130}$$

has no solution in the physical range $[0, 4M^2]$ [42] but always has a solution for $m^2_{a_1} > 4M^2$. Therefore the model fails to bind the $q\bar{q}$ pair in the a_1 channel because the on shell a_1 mass is above the mass gap $2M$. Even if the gradient expansion for the Lagrangian predicts a mass term, in practice we may not find it. In our method, after including pseudo-scalar, vector and pseudo-vector channels, it is believed that this problem will not be met. Of course further study is needed to show this explicitly.

Chapter 7

Conclusion

In this study, we have investigated some non-perturbative aspects of mass generation in Quantum Field Theory (QFT). We have started by introducing the Schwinger-Dyson equations as a tool to investigate QFT in a non-perturbative way. As they are an infinite system of equations, we introduced the idea of truncation, which is necessary to treat the equations in practice. We have written the equations satisfied for the electron and photon propagators in the bare vertex approximation as well as introduced our non-perturbative truncation scheme which respects multiplicative renormalisability (MR). We then presented the numerical method to solve systems of integral equations. The method is based on an expansion in Chebyshev polynomial and the Newton method to solve non-linear system of algebraic equations. Even though, we do not actually solve an algebraic system, the method is very efficient for our purpose.

In chapter three, we applied this model to the case of renormalised QED in the MR scheme and solved the equations in different approximations. For quenched QED, we found the usual critical coupling $\alpha_c = \pi/3$ and also showed the interesting result that in the Feynman gauge $\xi = 1$, it was possible to derive differential equations that do not stem from an Euclideanisation of the action and are therefore valid on the whole $x = p^2$ line. This simple result is an incentive to study further the relation between Minkowski and Euclidean formulations. We also determined the critical couplings in the one-loop approximation for the running coupling for a number of flavours N_f equal one and two, and solved the complete system by allowing the running coupling to satisfy its own equation.

After treating the case of QED, we proceeded to the quark equation and introduced a new truncation that is similar to the one of QED and to the one recently introduced by J. Bloch in [4]. The notable difference is that our truncation scheme makes the mass function explicitly independent of the renormalisation scale μ^2, as it should be. Using a model for the running coupling, we solved the equation for the mass function $M(p^2)$ and the dressing function $Z(p^2, \mu^2)$ and compared it to previous studies [4].

The study of bound state masses using the SD equations have already been undertaken by many authors. In this study, we used a simpler model, namely the Nambu Jona-Lasinio model to determine the masses of the ρ and π mesons as a function of four parameters: two couplings, a cut-off and the bare quark mass. In usual lattice calculations, meson masses are calculated for heavy quark masses and then extrapolated to small masses by using empirical methods such as cubic fit. As the NJL model is able to reproduce the pion mass correctly for any value of the bare quark mass, then it should be used to extrapolate lattice calculations for the pion. However we have seen that the NJL model is unable to reproduce the right quark dependence for the ρ meson mass, even when one tries to introduce another parameter.

Finally, as an outlook we have introduced a new method based on the control of quantum fluctuations by varying a bare parameter of the Lagrangian. If this parameter is multiplied by a quadratic operator, then it is possible to derive an integro-differential equation for the effective action. This equation is exact but needs truncation to be solved. The most promising approximation so far is to expand the effective action in a gradient expansion, which is believed to be more reliable than a gradient expansion of the Lagrangian as is usual in bosonisation methods, since the effective action is more localised in space than the Lagrangian. Another outlook, would be to inprove the numerical method. The system of equations for QED involved the equation for the coupling function, which had to be treated differently. The Newton method was only applied to the sub-system for the mass and dressing function, while we used an iteration process to determine the coupling function. A method in which all functions would be determined using the Newton method only would therefore be welcomed and would bring us more exact results. We could also try to find a way to rewrite the SD equations as a purely algebraic system of non-linear equation i.e. the unknowns coefficients should be multiplied by constants . A possible way is the following. Suppose we would like to solve the integral equation

$$m(x) = \int dy\, g(y, m(y))\, K(x,y)\,, \tag{7.1}$$

where $K(x,y)$ is the kernel of the equation and $g(y,m(y))$ is a known expression. Instead of expanding the function $m(x)$ on the basis of Chebyshev polynomial, we expand the function

$$z(y) = g(y, m(my)) = \sum{}' a_j T_j \tag{7.2}$$

which satisfies

$$m(x) = \int dy\, z(y)\, K(x,y)\,. \tag{7.3}$$

The solution $z(x)$ is obtained by solving

$$z(x) = g(x, m(x)) = g\left(x, \int dyz(y)\, K(x,y)\right)\,, \tag{7.4}$$

which after using Eq.(7.2) can be written

$$z(x) = g\left(x, \sum{}' a_j v_j(x,y)\right)\,, \tag{7.5}$$

with

$$v_j(x,y) = \int dy\, T_j\, K(x,y)\,. \tag{7.6}$$

The system of non-linear equations derived from Eq.(7.4) is now purely algebraic since the $v_j(x,y)$ can be computed once and for all. Once the function $z(x)$ is determined we can compute $m(x)$ for any value of x using Eq.(7.3) or as it could happen by inverting the relation Eq.(7.2), which would give $m(x)$ as a function of $z(x)$.

Our ultimate goal is the study of dynamical mass generation in QCD. We have shown that our truncation scheme permitted the factoring out of the renormalisation constants Z_3 and $\tilde{Z}_3$, when one treats the gluon-ghost sector at one loop. It remains to be checked that this is still so with the complete system by including the contributions from the three loop, four loop and quark diagrams. On the same occasion, an infrared study of this system would confirm or infirm the existence of an infrared fixed point for the coupling.

Finally, the whole approach of SD equations should maybe be abandonned to find a new and more powerful investigation tool. We hope that the functional approach or a similar one such as the exact renormalisation group approach can contribute to the advancement of our knowledge of infrared QCD.

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